THE JOURNAL

OF THE

INDIAN MATHEMATICAL SOCIETY

NEW SERIES
(*Issued Quarterly*)

Vol. II. No. 7

1937

PRINTED AT THE MADRAS LAW JOURNAL PRESS, MYLAPORE, MADRAS

Annual Subscription: Rs. 6] [Single Copy: Rupee One and Annas Eight

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ON A NET OF TETRAHEDRA ASSOCIATED WITH A SPACE CUBIC CURVE

By K. RANGASWAMI, M.A., M.Sc.

[Received 25 April. 1937]

1. It is well known that the locus of a point the feet of the perpendiculars from which to the faces of a tetrahedron lie in a plane is a quadrinodal cubic surface, having nodes at the vertices of the tetrahedron and containing its six edges. This locus* known as the Steiner's pedal locus of the tetrahedron is the isogonal transformation of the plane at infinity with respect to the tetrahedron and contains ∞^2 twisted cubics through the four nodes which are the transforms of the lines at infinity. Among these, *the cubics that are the transforms of the tangents to the absolute conic Ω have the peculiar property of meeting the plane at infinity in the vertices of a triangle circumscribed to the absolute conic.*

A special feature of any twisted cubic of this type, namely, that the feet of the perpendiculars from any point of the cubic to the faces of any inscribed tetrahedron lie in a plane was noticed by H. W. Richmond† who also extended the result to a norm-curve in an Euclidean space of n-dimensions. However, the converse problem of specifying those inscribed tetrahedra of a twisted cubic whose corresponding pedal locus contains the cubic, appears not to have been attempted so far. If for simplicity we call an inscribed tetrahedron Δ of a twisted cubic Γ a *pedal tetrahedron* if the feet of the perpendiculars from *any* point of the cubic to the faces of the tetrahedron lie in a plane, the main purpose of this paper is to show that *the totality of pedal tetrahedra of a twisted cubic constitutes a net.* It will be found that Richmond's result appears as a special case of our main theorem.

2. Let a, b, c, d be the lines of intersection of the plane at infinity with the faces of Δ and t a tangent to the absolute conic Ω.

* Baker, *Principles of Geometry*, Vol. IV, 12-27.

† *Proc. Camb. Phil. Soc.* 22, 34-38.

Consider the net (N) of quadrics apolar* to Δ and outpolar to Ω. The quadrics of this net will meet the plane at infinity in a net (n) of conics which will be outpolar to Ω and also to any inconic of the quadrilateral δ formed by the lines a, b, c, d. Hence the three pairs of opposite vertices of any quadrilateral circumscribed to Ω and to any inconic of δ will be degenerate class conics inpolar to the conics of the net (n) and therefore will be isogonal conjugates† with respect to Δ. It follows therefore that if S is the unique conic touching a, b, c, d, t and if t_1, t_2, t_3 be the three common tangents of S and Ω other than t, the isogonal transformation of the line t will be a twisted cubic through the vertices of Δ and the points (t_2, t_3), (t_3, t_1), (t_1, t_2). We have thus the result:

The isogonal transformation af any tangent to the absolute conic is a twisted cubic through the vertices of Δ which meets the plane at infinity in the vertices of a triangle circumscribed to the absolute conic. (2.1)

Now, the seven linearly independent class quadrics inpolar to the net (N) of order quadrics may be taken to be the six pairs of points consisting of any two vertices of Δ and the absolute conic Ω. Hence any class quadric inpolar to the net (N) must be linearly dependent on these. Thus, a pair of points p, p' conjugate to all the quadrics of the net, i.e. a pair of isogonal conjugates for Δ, must belong to this linear system and hence must be the foci of a quadric of revolution‡ inscribed in Δ. In particular if p' is at infinity one of the quadrics of the net (N) has its centre at p, and in this case p is the finite focus of an inscribed paraboloid of revolution. Consequently the feet of the perpendiculars from p on the faces of Δ lie in a plane. Hence we see that:

If the feet of the perpendiculars from p on the faces of Δ lie on a plane then p is the centre of a quadric of the net (N).

3. Let P, Q, R be the points in which the twisted cubic Γ meets the plane at infinity and Δ any inscribed tetrahedron. Now in order that a point O on Γ should have a pedal plane for Δ it is

* The term 'apolar' is here used to mean 'having Δ for a self-polar tetrahedron',

† Two points are said to be isogonal conjugates with respect to a tetrahedron Δ when they are conjugate with respect to the net of quadrics having Δ for a self-polar tetrahedron and outpolar to the 'Absolute'.

‡ If the tangential equations of the vertices of Δ be $A=0$, $B=0$, $C=0$, $D=0$, those of a pair of isogonal conjugates be $P=0$, $P'=0$ and the tangential equation of the absolute conic be $\Omega=0$, there is a relation of the form

$$\lambda PP'+\Omega=\lambda_{AD}AD+\ldots+\lambda_{BC}BC+\ldots .$$

necessary and sufficient that the unique order-quadric apolar to the two tetrahedra Δ and $OPQR$ must be also outpolar to Ω. Taking the familiar mode of representing the inscribed tetrahedra* of a twisted cubic in [3] by the points of [4], the quadrics inpolar to Γ and outpolar to Ω will be represented by the lines of a linear complex in [4]. The unique tetrahedron which is in general inscribed in Γ and circumscribed to Ω corresponds to the singular point of the linear complex. Since $OPQR$ is a pedal tetrad on Γ, by varying O on Γ, we have a pencil of pedal tetrads which is represented in [4] by a line l of the linear complex.

Now, it is easily seen that if Δ is a pedal tetrad, any two points and therefore every point on Γ should have pedal planes for Δ. Thus by our representation the problem of determining the pedal tetrads on Γ is equivalent to that of finding the points Δ in [4] the line joining which to any point on l is a complex line. It follows, therefore, that all such points lie in the polar plane of l with respect to the linear complex which is also the plane determined by l and the singular point ω of the linear complex. We have thus proved the main result of this paper, viz.:

The totality of pedal tetrads on a cubic curve Γ *constitutes a net.* (3.1)

We note further that if Δ be any tetrahedron of this net, the pencil of tetrahedra determined by Δ and $OPQR$ will also be pedal tetrads.

Proceeding to the case when the triangle PQR is circumscribed to the absolute conic Ω, every tetrahedron $OPQR$ is circumscribed to Ω, so that there is a pencil of inscribed tetrahedra of Γ circumscribed to Ω. Hence by a known theorem† there must be a net of inscribed tetrahedra of Γ also circumscribed to Ω. Consequently the linear complex in [4] has a plane ω of singular points so that the line joining *any* point Δ of [4] to any point of the singular plane is necessarily a complex line. Hence in this case every tetrad of Γ is a pedal tetrad—which is Richmond's result stated in the beginning.

4. We will now proceed to specify geometrically the net of pedal tetrads on the cubic curve Γ. Now if Δ is a pedal tetrahedron on Γ, it is easily seen from (2.2) that among the quadrics of the net (N) there is a pencil whose centres lie on Γ.

* R. Vaidyanathaswamy, 'On the rational Norm curve II', *Jour. Lond. Math. Soc.* 7, 54-5.

† On the rational Norm curve II, loc. cit. p. 54.

The quadrics of this pencil meet the plane at infinity in a pencil γ of conics contained in the net (n) and having PQR for a self-polar triangle. Since the quadrics of the net (N) are apolar to Δ the three pairs of opposite edges of Δ meet the plane at infinity in three pairs of points conjugate for the conics of the net (n) and therefore for the conics of the pencil γ. We thus infer that the pairs of opposite edges of Δ meet the plane at infinity in pairs of isogonal conjugate points* with respect to the triangle PQR.

Conversely, let p, p' be a pair of isogonal conjugate points with respect to the triangle PQR and Δ the inscribed tetrahedron formed by drawing the chords of Γ through p, p'. Now it is easily seen from the characteristic property of the twisted cubic, that QR, RP, PQ determine related ranges with the faces of Δ. Hence if the faces of Δ meet the plane at infinity in the lines a, b, c, d there is a definite conic σ touching a, b, c, d and also the sides of the triangle PQR. Thus the pencil γ being outpolar to the linearly independent conics σ, Ω and (p, p') is contained in the net (n). This implies that there is a pencil of quadrics of the net (N). whose centres lie on Γ; in other words, Δ is a pedal tetrahedron of Γ. We have therefore the result:

The vertices of a pedal tetrahedron are the extremities of chords of Γ *through a pair of isogonal conjugate points with respect to the triangle PQR.* (4.1)

* 'Isogonal conjugates' in the non-euclidean plane with Ω as the absolute conic.

THE MULTIPLICATIVE ARITHMETIC FUNCTIONS CONNECTED WITH A FINITE ABELIAN GROUP

By T. VENKATARAYUDU, M.A., University of Madras

Introduction.

An arithmetic function $f(N)$ is multiplicative* if

$$f(MN)=f(M)f(N)$$

whenever the integers M, N are relatively prime. The function $F(N)$ defined by the equation

$$F(N)=\sum f_1(\delta)f_2\left(\frac{N}{\delta}\right)$$

summed for all divisors δ of N is called the *composite* of the two arithmetic functions f_1, f_2. If f_1 and f_2 are multiplicative, it is easy to see that their composite F (represented by $f_1.f_2$) is also multiplicative.

Analogous definitions may be given for functions of several arguments. Thus the arithmetic function $f(M_1, M_2, \ldots M_r)$ is said to be multiplicative if

$$f(M_1N_1, M_2N_2, \ldots M_rN_r)=f(M_1, M_2, \ldots M_r)f(N_1, N_2, \ldots N_r)$$

whenever the products $M_1M_2\ldots M_r$, $N_1N_2\ldots N_r$ are relatively prime. The notion of composition may be obviously extended to functions of several arguments.

The function of r arguments $M_1, M_2, \ldots M_r$ which takes the value 1 for all values of its arguments is denoted by $E(M_1, M_2 \ldots M_r)$, The composite of $f(M_1, M_2, \ldots M_r)$ and $E(M_1, M_2, \ldots M_r)$

is given by $\quad (f{\cdot}E)(M_1, M_2, \ldots M_r)=\Sigma f(\delta_1, \delta_2, \ldots \delta_r)$

summed for all divisors δ_i of M_i $(i=1, 2, \ldots r)$. We call $f{\cdot}E$ as the *integral* of f.

It is known that given an arithmetic function $f(N)$ there exists a unique arithmetic function $\psi(N)$ such that the composite $f.\psi$ vanishes for all values of its argument other than 1 and takes

* *See* Dr. R. Vaidyanathaswamy "The theory of multiplicative arithmetic functions" *Trans. Amer. Math. Soc.*, Vol. 33. pp. 579-662. (1931).

the value 1 when the argument is equal to 1. We call ψ the *inverse function* of f and denote it by f^{-1}. It is easy to see that f^{-1} is also a multiplicative function.

The multiplicative arithmetic function $f(N)$ is said to be a *rational integral function* of degree r if $f^{-1}(N)$ vanishes for all values of N divisible by an $(r+1)$th power.

In the present paper we consider the four multiplicative arithmetic functions $m(d_1, d_2)$, $m(d)$, $M(d)$, $\Gamma(d_1, d_2, \ldots d_r)$ (defined below) connected with a finite Abelian group.

I. The multiplicative functions.

Let G be an Abelian group of finite order N. We require the known*

LEMMA: An element S of order d_1d_2 in G where $(d_1, d_2)=1$ can be uniquely expressed as the product of two elements of orders d_1 and d_2 respectively.

PROOF: If $$S^{d_2}=M,\ S^{d_1}=N$$
and $$ad_1+bd_2=1$$
M^b, N^a are of orders d_1 and d_2. Then $S=M^b N^a$ is the required expression.
If possible, let $S=M_1N_1$ where M_1, N_1 are of orders d_1 and d_2.
$$S^{d_2}=M_1^{d_2}=M.$$
Therefore $$M^b=M_1^{bd_2}=M_1^{1-ad_1}=M_1.$$
Similarly $N^a=N_1$. Hence the representation is unique.

We denote by $G(d_1, d_2)$ the totality of the elements in G whose orders divide d_1 and are multiples of d_2. Let $m(d_1, d_2)$ denote the number of elements in $G(d_1, d_2)$.

THEOREM 1: $m(d_1, d_2)$ is a multiplicative arithmetic function of the two arguments d_1, d_2.

PROOF: Let the product $G(d_1, d_2).G(d'_1, d'_2)$ represent the totality of the elements obtained by multiplying each element of $G(d_1, d_2)$ with each element of $G(d'_1, d'_2)$. We first show that
$$G(d_1, d_2).G(d'_1, d'_2)=G(d_1d'_1, d_2d'_2) \tag{1}$$
whenever $(d_1d_2, d'_1d'_2)=1$.

Evidently $G(d_1, d_2).G(d'_1, d'_2)$ is contained in $G(d_1d'_1, d_2d'_2)$. Again if S is an element in $G(d_1d'_1, d_2d'_2)$ of order d we

* Burnside. Theory of groups (1911), pp. 15-16.

can write $d=\delta\delta'$ uniquely where δ is a divisor of d_1 and a multiple of d_2 and δ' is a divisor of d'_1 and a multiple of d'_2. By the lemma we can express S as S_1S_2 where S_1 is of order δ and S_2 is of order δ'. Thus $G(d_1d'_1, d_2d'_2)$ is contained in the product $G(d_1, d_2).G(d'_1, d'_2)$. Hence (1) is established and therefore, $m(d_1, d_2)m(d'_1, d'_2)=m(d_1d'_1, d_2d'_2)$* whenever $(d_1d_2, d'_1d'_2)=1$.

THEOREM 2. If $G(d)$ denotes the totality of the elements in G whose orders divide d and $m(d)$ is the number of elements in $G(d)$, $m(d)$ is a multiplicative arithmetic function.

PROOF: This follows immediately from Theorem 1 for

$$m(d)=m(d, 1)\dagger$$

THEOREM 3. If $C(d)$ is the class of elements of order d in G and if $M(d)$ denotes the number of elements in $C(d)$, $M(d)$ is a multiplicative function.

PROOF: This can be proved directly by showing

$$C(dd')=C(d)C(d') \quad \text{when } (d, d')=1.$$

or thus:

$$m(d)=\Sigma M(\delta) \text{ summed for all divisors } \delta \text{ of } d.$$

$m(d)$ is therefore the integral of $M(d)$. Since $m(d)$ is multiplicative so also is $M(d)$.

It is well known‡ that every Abelian group is the direct product of cyclic groups of prime power orders. Let a be the greatest of the exponents of the prime powers. $m(d), M(d)$ vanish for values of d divisible by an $(a+1)$th power and therefore $m(d)$, $M(d)$ are inverses of rational integral functions of degree a. Also $m(d)=0=M(d)$ if d is prime to N.

I showed elsewhere‖ that the classes $C(d)$ combine among themselves by multiplication. That is, if $C(d_i)C(d_j)$ represents the totality of the elements obtained by multiplying each element of $C(d_i)$ with each element of $C(d_j)$ then each element of $C(d_k)$ occurs the same number of times $\gamma(d_i, d_j, d_k)$ in the product $C(d_i)$ $C(d_j)$. Let $\Gamma(d_1, d_2, \ldots d_r)$ denote the number of times the identity class (=element) occurs in the product $C(d_1)C(d_2)\ldots C(d_r)$

* Since $m(d_1, d_2)$ vanishes whenever d_2 is not a divisor of d_1 it is an *ordinal function*. *See* Dr. R. Vaidyanathaswamy loc. cit. p. 656.

† $m(d)$ is the derivative $D_{d_2}[m(d_1, d_2)]$ *Cf.* Dr. R. Vaidyanathaswamy, loc. cit., p. 623.

‡ Speiser "Theorie der Gruppen von Endlicher ordnung", p. 50.

‖ "On the linear algebra of classes of elements in a finite Abelian group", under publication in the *Proc. Ind. Acad. Sc.*

and $\gamma(d_1, d_2 \ldots d_r)$ denote the number of times the class $C(d_r)$ occurs in the product $C(d_1).C(d_2) \ldots C(d_{r-1})$. Since the inverses of the elements in a class $C(d)$ form the class $C(d)$ itself it is easy to see that

$$M(d_r).\gamma(d_1, d_2 \ldots d_r) = \Gamma(d_1, d_2, \ldots d_r).$$

The function $\Gamma(d_1, d_2, \ldots d_r)$ is clearly symmetric and we have

THEOREM 4. $\Gamma(d_1, d_2, \ldots d_r)$ is a symmetric multiplicative function of the r arguments $d_1, d_2, \ldots d_r$.

PROOF: By definition $\Gamma(d_1d'_1, d_2d'_2, \ldots d_rd'_r)$ = the number of times the identity element occurs in the product $C(d_1d'_1)C(d_2d'_2) \ldots C(d_rd'_r)$. If the products $d_1d_2 \ldots d_r$, $d'_1d'_2 \ldots d'_r$ are relatively prime $\Gamma(d_1d'_1, d_2d'_2, \ldots d_rd'_r)$ = the number of times the identity occurs in

$$C(d_1)C(d_2) \ldots C(d_r).C(d'_1)C(d'_2) \ldots C(d'_r).$$

i.e. in

$$\left\{ \Sigma\gamma(d_1, d_2, \ldots d_r, d_k)C(d_k). \right\}\left\{ \Sigma\gamma(d'_1, d'_2, \ldots d'_r, d'_k)C(d'_k) \right\}$$
$$= \gamma(d_1, d_2, \ldots d_r, 1).\gamma(d'_1, d'_2, \ldots d'_r, 1).$$
$$= \Gamma(d_1, d_2, \ldots d_r).\Gamma(d'_1, d'_2, \ldots d'_r).$$

Hence $\Gamma(d_1, d_2, \ldots d_r)$ is a multiplicative function.

Cor. $\gamma(d_1, d_2, \ldots d_r)$ is multiplicative.

II. EXPRESSIONS FOR THE MULTIPLICATIVE FUNCTIONS.

Since $m(d)$, $M(d)$, $m(d_1, d_2)$, $\Gamma(d_1, d_2, \ldots d_r)$ are multiplicative it is enough if we find their values when the arguments are powers of a single prime p. Hence the functions depend only on the sub-group of G, consisting of all elements in G whose orders are powers of p (the sylow component of G belonging to the prime p). Let the type of this group be $(\alpha_1 \geqslant \alpha_2 \geqslant \ldots \geqslant \alpha_r)$.

$m(p^\beta) = p^{k\beta + \alpha_{k+1} + \ldots + \alpha_r}$ where k is the number of α's $\geqslant \beta$* (1)

Therefore $M(p^\beta) = p^{k(\beta-1) + \alpha_{k+1} + \ldots + \alpha_r}(p^k - 1)$ (2)

From (1) and (2) the value of $m(p^{a_1}, p^{a_2})$, $a_1 > a_2$ can be obtained

$$m(p^{a_1}, p^{a_2}) = m(p^{a_1}) - m(p^{a_2}) + M(p^{a_2}) \qquad (3)$$

Expression for $\gamma(p^{a_i}, p^{a_j}, p^{a_k})$:—I have shown that

$$\gamma(d_i, d_j, d_k) = \sum m[g(t_i, t_j)]\mu\left(\frac{d_i}{t_i}\right)\mu\left(\frac{d_j}{t_j}\right)$$

* Miller, Blichfeldt, and Dickson "Finite groups", p. 93.

summed for divisors t_i, t_j of d_i, d_j such that the l.c.m. of t_i. t_j is a multiple of d_k. If we take $d_i=p^{a_i}$, $d_j=p^{a_j}$, $d_k=p^{a_k}$ we have the following cases:

(1) When $a_i>a_j$; $a_i>a_k$

$$\gamma(p^{a_i}, p^{a_j}, p^{a_k})=0.$$

(2) When $a_i>a_j$; $a_i=a_k$

$$\gamma(p^{a_i}, p^{a_j}, p^{a_k})=m(p^{a_j})-m(p^{a_j-1})=M(p^{a_j}).$$

(3) When $a_i=a_j$; $a_i>a_k$

$$\gamma(p^{a_i}, p^{a_j}, p^{a_k})=m(p^{a_i})-m(p^{a_i-1})=M(p^{a_i}).$$

(4) When $a_i=a_j=a_k$

$$\gamma(p^{a_i}, p^{a_j}, p^{a_k})=m(p^{a_i})-2m(p^{a_i-1})$$

$$=M(p^{a_i}).\left[1-\frac{m(p^{a_i-1})}{M(p^{a_i})}\right].$$

We can now state the value of $\gamma(d_i, d_j, d_k)$ in the general case viz.

$$\gamma(d_i, d_j, d_k)=M[g(d_i, d_j)]\Pi\left[1-\frac{m(p^{a-1})}{M(p^a)}\right], \text{ or } 0$$

according as the l.c.m. of every two of d_i, d_j, d_k is the same or not and the product extending over the common block factors* of d_i, d_j, d_k. $\gamma(p^{a_i}, p^{a_j}, p^{a_k})$ can be directly obtained by a method which we use below but the above form suggests an important class of multiplicative arithmetic functions.

Expression for $\Gamma(p^{a_1}, p^{a_2},\ldots p^{a_r})$.

Since $\Gamma(p^{a_1}, p^{a_2},\ldots p^{a_r})$ is symmetric we can take

$$a_1\geqslant a_2\geqslant\ldots\geqslant a_r.$$

$$C(p^{a_1})C(p^{a_2})\ldots C(p^{a_r})=$$

$$\left\{G(p^{a_1})-G(p^{a_1-1})\right\}\left\{G(p^{a_2})-G(p^{a_2-1})\right\}\ldots$$
$$\times\left\{G(p^{a_r})-G(p^{a_r-1})\right\}$$

Case i. $a_1\neq a_2$. $C(p^{a_2})$, $C(p^{a_3})\ldots C(p^{a_r})$ are complexes (See my paper loc. cit.) in $G(p^{a_1})$ and $G(p^{a_1-1})$. Hence

$$C(p^{a_1}).C(p^{a_2})\ldots C(p^{a_r})=M(p^{a_2}).M(p^{a_3})\ldots M(p^{a_r}).C(p^{a_1}).$$

* My paper, loc. cit.

Case ii, $a_1=a_2=\ldots=a_k=a\neq a_{k+1}$.

In this case $C(p^{a_1}).C(p^{a_2})\ldots C(p^{a_r})$

$$=M(p^{a_{k+1}})M(p^{a_{k+2}})\ldots M(p^{a_r}).\left\{G(p^a)-G(p^{a-1})\right\}^k$$

$$=M(p^{a_{k+1}})\ldots M(p^{a_r}).\Big[G(p^a)-\binom{k}{1}G(p^a)^{k-1}G(p^{a-1})+\ldots$$

$$+(-1)^{k-1}\binom{k}{k-1}G(p^{a-1})^{k-1}G(p^a)+(-1)^kG(p^{a-1})^k\Big]$$

$$=M(p^{a_{k+1}})\ldots M(p^{a_r}).\Big[m(p^a)^{k-1}G(p^a)$$

$$-\binom{k}{1}m(p^a)^{k-2}.\,m(p^{a-1}).G(p^a)+\ldots$$

$$+(-1)^{k-1}km(p^{a-1})^{k-1}G(p^a)+(-1)^k m(p^{a-1})^{k-1}.G(p^{a-1})\Big].$$

$$=M(p^{a_{k+1}})\ldots M(p^{a_r}).\Big[\frac{\{m(p^a)-m(p^{a-1})\}^k+(-1)^{k-1}m(p^{a-1})^k}{m(p^a)}$$

$$\times G(p^a)+(-1)^k m(p^{a-1})^{k-1}G(p^{a-1})\Big]$$

$$=M(p^{a_{k+1}})\ldots M(p^{a_r}).\Big[\frac{M(p^a)^k+(-1)^{k-1}m(p^{a-1})^k}{m(p^a)}G(p^a)$$

$$+(-1)^k m(p^{a-1})^{k-1}G(p^{a-1})\Big]$$

Hence $\Gamma(p^{a_1}, p^{a_2},\ldots p^{a_r})=0$ if $a_1\neq a_2$,

$$=M(p^{a_{k+1}})\ldots M(p^{a_r}).\Big[\frac{M(p^a)^k+(-1)^{k-1}.m(p^{a-1})^k}{m(p^a)}$$

$$+(-1)^k.m(p^{a-1})^{k-1}\Big]$$

if $a=a_1=\ldots=a_k\neq a_{k+1}$.

My thanks are due to Dr. R. Vaidyanathaswamy for his kind help with the manuscript.

ON SOME THEOREMS CONCERNING DETERMINANTAL SYMMETRIC FUNCTIONS

By M. ZIA-UD-DIN, M.A.Ph.D. (Wales), Bahawalpur

[Received 26 April 1937]

1. In a previous paper* a fundamental theorem was given by the author by means of which the general isobaric determinants are expressed as a sum of bi-alternant symmetric functions. In this paper I shall deal with the generalisation of the theorem and certain deductions that follow from it. The *theorem* is:

If the bi-alternant $h\begin{pmatrix} 0 & p & q\ldots \\ 0 & 1 & 2\ldots \end{pmatrix}$ be expanded in terms of monomial symmetric functions $\Sigma\, a^i b^j c^k \ldots$, as

$$h\begin{pmatrix} 0 & p & q\ldots \\ 0 & 1 & 2\ldots \end{pmatrix} = \ldots + \rho_{ijk\ldots} P(ijk\ldots) + \ldots;\text{ then}$$

$$h\begin{pmatrix} \alpha & \beta & \gamma\ldots \\ 0 & p & q\ldots \end{pmatrix} = \ldots + \rho_{ijk\ldots} \sum_{ijk\ldots} h\begin{pmatrix} \alpha-i & \beta-j & \gamma-k\ldots \\ 0 & 1 & 2\ \ldots \end{pmatrix}$$

where Σ indicates permutation of $i, j, k, \ldots$ and summation, $\alpha, \beta, \gamma, \ldots$; $p, q, r, \ldots$ are in ascending order of magnitude and the expansion being indicated by a single typical term.

2. Before proceeding to the generalisation of the above theorem, a theorem due to D. E. Littlewood will be deduced here.

Following Littlewood and Richardson,† we have

$$\{\lambda/\mu\} = \left| {}^h\lambda_s - \mu_t - s + t \right| = h\begin{pmatrix} \lambda_1+n-1 & \lambda_2+n-2\ldots\lambda_{n-1}+1 & \lambda_n \\ \mu_1+n-1 & \mu_2+n-2\ldots\mu_{n-1}+1 & \mu_n \end{pmatrix}$$

where (λ) denotes the partition $(\lambda_1, \lambda_2, \ldots \lambda_n)$ and (μ) the partition $(\mu_1, \mu_2, \ldots \mu_n)$, $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n$, $\mu_1 \geqslant \mu_2 \geqslant \ldots \geqslant \mu_n$; s denotes the row and t the column of the matrix from which the element is taken, the determinant having n rows and columns; and

$$\{\lambda\} = \left| {}^h\lambda_s - s + t \right| = \frac{A(\alpha\ \beta\ \gamma\ldots)}{A(0\ 1\ 2\ldots)}.$$

By taking $\mu_n = 0$, we can write

* Zia-ud-Din, *Proc. Edin. Math. Soc.* (2) 4 (1934) 47-52.
† *See* D. E. Littlewood, *Proc. London, M. S.* (2) 40 (1936) 60.

$$\{\lambda/\mu\} = h\begin{pmatrix} \lambda_n & \lambda_{n-1}+1 & \lambda_{n-2}+2 \ldots \lambda_1+n-1 \\ 0 & \mu_{n-1}+1 & \mu_{n-2}+2 \ldots \mu_1+n-1 \end{pmatrix}$$

$$= \ldots + \rho_{ijk\ldots} \sum_{ijk\ldots} h\begin{pmatrix} \lambda_n - i & \lambda_{n-1}+1-j & \lambda_{n-2}+2-k \ldots \\ 0 & 1 & 2 \quad \ldots \end{pmatrix}$$

Put $$h\begin{pmatrix} \lambda_n - i & \lambda_{n-1}+1-j \ldots \\ 0 & 1 \quad \ldots \end{pmatrix} = \{\nu\} \text{ and}$$

$$\Sigma\, \rho_{i'j'k'\ldots} P(i'j'k'\ldots) = \{\mu\},$$

(Σ denoting summation for different values of i', j', k',..., including i, j, k,...).

Since*

$$A(\alpha_1\beta_1\gamma_1\ldots)P(\alpha_2\beta_2\gamma_2\ldots) = A(\alpha_1 + \alpha_2^{+}, \beta_1 + \beta_2^{+}, \gamma_1 + \gamma_2^{+}, \ldots)$$

therefore the coefficient of

$$A(\lambda_n \lambda_{n-1}+1 \ldots \lambda_1+n-1) \text{ in}$$

$A(\lambda_n - i\ \lambda_{n-1}+1-j\ \lambda_{n-2}+2-k\ldots)\Sigma\, \rho_{i'j'\ldots}P(i'j'k'\ldots)$, that is in $\{\nu\}A(012\ldots)\{\mu\}$ is clearly $\rho_{ijk\ldots}$. Hence $\rho_{ijk\ldots}$ is the coefficient of $\{\lambda\}$ in the product $\{\nu\}\{\mu\}$.

Similarly by assigning different values to $i, j, k, \ldots$ in $\{\nu\}$, the coefficient of $\{\lambda\}$ in $\{\nu\}\{\mu\}$ will evidently be $\rho_{ijk\ldots}$ according to those values.

Hence we obtain Littlewood's† Theorem

$$\{\lambda/\mu\} = \left| {}^h\lambda_s - \mu_t - s + t \right| = \sum g_{\nu\mu\lambda}\{\nu\}$$

where $g_{\nu\mu\lambda}$ is the coefficient of $\{\lambda\}$ in the product $\{\nu\}\{\mu\}$.

3. The theorem of expressibility of general isobarics as a sum of simple-isobarics or S-functions can be generalised as

THEOREM. *To a rational integral identity of t variables.*

$$A(\lambda\mu\nu\ldots) = A(lmn\ldots)P(pqr\ldots) \pm A(l'm'n'\ldots)P(p'q'r'\ldots) + \ldots, \qquad I$$

where $(l+m+n+\ldots)+(p+q+r+\ldots) = \lambda+\mu+\nu+\ldots,$

* Zia-ud-Din *l.c.* p. 49.

N.B. $A(\alpha\beta\gamma\ldots)$ denotes the Alternant $|a^\alpha b^\beta c^\gamma \ldots|$ and the associated permanent which forms the monomial symmetric functions $\Sigma a^\alpha b^\beta c^\gamma\ldots$, is denoted by $P(\alpha\beta\gamma\ldots)$ and it will be termed as *Monomial Permanent.*

† D. E. Littlewood, *l.c.* p. 61, Theorem VIII.

$\lambda, \mu, \nu \ldots, l, m, n, \ldots$ etc being in ascending order of magnitude, there corresponds the identity in isobaric determinants of t-th order,

$$h\begin{pmatrix} \alpha & \beta & \gamma & \ldots \\ \lambda & \mu & \nu & \ldots \end{pmatrix} = \sum h\begin{pmatrix} \alpha-p & \beta-q & \gamma-r & \ldots \\ l & m & n & \ldots \end{pmatrix}$$

$$\pm \sum h\begin{pmatrix} \alpha-p' & \beta-q' & \gamma-r' & \ldots \\ l' & m' & n' & \ldots \end{pmatrix} + \ldots \qquad II$$

where Σ denotes permutation of $p, q, r, \ldots$ etc. and summation.

Proof. In the alternants and monomial symmetric functions, use, instead of $a, b, c, \ldots$ the operational letters $A, B, C, \ldots$ which are such that $A^k h_\alpha = h_{\alpha-k}$, $B^k h_\beta = h_{\beta-k}$, $C^k h_\gamma = h_{\gamma-k}$, and so on. Thus identity I can be operationally written as,

$$H(\lambda\mu\nu\ldots) = H(lmn\ldots)\Sigma A^p B^q C^r \ldots$$

$$\pm H(l'm'n'\ldots)\Sigma A^{p'} B^{q'} C^{r'} \ldots + \ldots$$

Performing the operations on $h_\alpha h_\beta h_\gamma \ldots$, we obtain the identity II,

$$h\begin{pmatrix} \alpha & \beta & \gamma & \ldots \\ \lambda & \mu & \nu & \ldots \end{pmatrix} = \sum_{pqr\ldots} h\begin{pmatrix} \alpha-p & \beta-q & \gamma-r & \ldots \\ l & m & n & \ldots \end{pmatrix}$$

$$\pm \sum_{p'q'r'\ldots} h\begin{pmatrix} \alpha-p' & \beta-q' & \gamma-r' & \ldots \\ l' & m' & n' & \ldots \end{pmatrix}$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

which proves the Theorem.

4. The identity II does not depend merely on the fact that h's are symmetric functions, but we could quite well use determinantal elements. Thus the identity in pure determinants corresponding to

$$A(026) = A(015)P(011) - A(014)P(111), \qquad (1)$$

can be written as,

$$\begin{vmatrix} a_0 & b_0 & c_0 \\ a_2 & b_2 & c_2 \\ a_6 & b_6 & c_6 \end{vmatrix} = \begin{vmatrix} a_0 & b_1 & c_1 \\ a_1 & b_2 & c_2 \\ a_5 & b_6 & c_6 \end{vmatrix} + \begin{vmatrix} a_1 & b_0 & c_1 \\ a_2 & b_1 & c_2 \\ a_6 & b_5 & c_6 \end{vmatrix}$$

$$+ \begin{vmatrix} a_1 & b_1 & c_0 \\ a_2 & b_2 & c_1 \\ a_6 & b_6 & c_5 \end{vmatrix} - \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_5 & b_5 & c_5 \end{vmatrix}$$

which can be verified on expansion.

Let the operation of increasing by k the suffixes in the s-th row of the determinant $|a_0 b_1 c_2 \ldots|$ of n-th order be denoted by E_s^k

and that of increasing by σ the suffixes in column t by F_t^σ. These operators will be called *Displacement Operators* as they correspond to moving rows and columns about.

Now taking the operand of 3rd order

$$\begin{vmatrix} a_0 & b_0 & c_0 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

the operational identity corresponding to (1) is

$$\begin{aligned} E_1^0E_2^1E_3^4 &= E_1^0E_2^0E_3^3(F_1^0F_2^1F_3^1+F_1^1F_2^0F_3^1+F_1^1F_2^1F_3^0) \\ &\quad -E_1^0E_2^0E_3^2(F_1^1F_2^1F_3^1), \\ &= E_1^0E_2^0E_3^3(\Sigma\, F_1^0F_2^1F_3^1)-E_1^0E_2^0E_3^2(F_1^1F_2^1F_3^1). \qquad (2) \end{aligned}$$

Generally corresponding to identity I of Art 3, between the products of alternants and monomial Permanents there is an identity in pure determinants,

$$E_1^\lambda E_2^{\mu-1}E_3^{\nu-2}\ldots = E_1^l E_2^{m-1}E_3^{n-2}\ldots\Big(\sum_{pqr\ldots} F_1^pF_2^qF_3^r\ldots\Big)+\ldots; \qquad III$$

when the operand is of the type.

$$|\, a_0 \quad b_1 \quad c_2 \ldots |.$$

4.1. As an example we can apply III to the Wronskians, say,

$$\begin{vmatrix} f_1 & f_2 & f_3\ldots \\ f'_1 & f'_2 & f'_3\ldots \\ f''_1 & f''_2 & f''_3\ldots \\ \ldots & \ldots & \ldots \end{vmatrix}$$

dashes denoting differentiations.

Thus (2) may be written as

$$\begin{vmatrix} f_1 & f_2 & f_3 \\ f''_1 & f''_2 & f''_3 \\ f_1^{\text{vi}} & f_2^{\text{vi}} & f_3^{\text{vi}} \end{vmatrix} = \begin{vmatrix} f_1 & f'_2 & f'_3 \\ f'_1 & f''_2 & f''_3 \\ f_1^{\text{v}} & f_2^{\text{vi}} & f_3^{\text{vi}} \end{vmatrix}$$

$$+\begin{vmatrix} f'_1 & f_2 & f'_3 \\ f''_1 & f'_2 & f''_3 \\ f_1^{\text{vi}} & f_2^{\text{v}} & f_3^{\text{vi}} \end{vmatrix} + \begin{vmatrix} f'_1 & f'_2 & f_3 \\ f''_1 & f''_2 & f'_3 \\ f_1^{\text{vi}} & f_2^{\text{vi}} & f_3^{\text{v}} \end{vmatrix}$$

$$-\begin{vmatrix} f'_1 & f'_2 & f'_3 \\ f''_1 & f''_2 & f''_3 \\ f_1^{\text{v}} & f_2^{\text{v}} & f_3^{\text{v}} \end{vmatrix}.$$

which again may be verified.

EXTENSIONS OF SOME SELF-RECIPROCAL FUNCTIONS

By R. S. VARMA

1. In a recent paper* I have shown that the functions

$$x^{\nu+\frac{1}{2}}e^{\frac{1}{4}x^2}D_{-2\nu-3}(x) \qquad R(\nu)>-1$$

and

$$x^{\nu-\frac{1}{2}}e^{\frac{1}{4}x^2}D_{-2\nu}(x) \qquad R(\nu)>-\tfrac{1}{2}$$

are self-reciprocal in the Hankel-transform of order ν, that is,

$$x^{\nu+\frac{1}{2}}e^{\frac{1}{4}x^2}D_{-2\nu-3}(x)=\int_0^\infty \sqrt{(xy)}J_\nu(xy)\,y^{\nu+\frac{1}{2}}e^{\frac{1}{4}y^2}D_{-2\nu-3}(y)\,dy. \qquad (1)$$

and

$$x^{\nu-\frac{1}{2}}e^{\frac{1}{4}x^2}D_{-2\nu}(x)=\int_0^\infty \sqrt{(xy)}J_\nu(xy)\,y^{\nu-\frac{1}{2}}e^{\frac{1}{4}y^2}D_{-2\nu}(y)\,dy. \qquad (2)$$

The object of the present paper is to amplify and generalise these results. Thus in § 2 of this paper I have investigated the integral

$$\int_0^\infty y^{n+1}J_n(xy)\,e^{\frac{1}{4}y^2}D_{-m}(y)\,dy. \qquad (3)$$

$$=\frac{x^n\Gamma(2n+2)}{2^{n+1}\Gamma(m)\Gamma(n+1)}\Big[2^{\frac{1}{2}m-n-1}\Gamma(\tfrac{1}{2}m-n-1)$$

$$\times{}_1F_1(n+\tfrac{3}{2};\,2+n-\tfrac{1}{2}m;\,\tfrac{1}{2}x^2)$$

$$+x^{m-2n-2}\frac{\Gamma(1+n-\frac{1}{2}m)\Gamma(\frac{1}{2}m+\frac{1}{2})}{\Gamma(n+\frac{3}{2})}\,{}_1F_1(\tfrac{1}{2}m+\tfrac{1}{2};\,-n+\tfrac{1}{2}m;\,\tfrac{1}{2}x^2)\Big]$$

$$R(m)>0,\ R(n+1)>0 \text{ and } R(n-m+\tfrac{1}{2})<0$$

which gives the generalisation of the integral equation (1). The extension of the integral equation (2) is furnished by the following integral, deduced by me elsewhere† by a different method,

* R. S. Varma, "Some functions which are self-reciprocal in the Hankel-transform", *Proc. Lond. Math. Soc.* (2) 42 (1936), 9-17.

† R. S. Varma, "An infinite integral involving Bessel functions and parabolic cylinder functions", *Proc. Camb. Phil. Soc.* 33 (1937), 210-11.

$$\int_0^\infty y^{n-\frac{1}{2}}e^{\frac{1}{4}y^2}J_{n-\frac{1}{2}}(xy)D_{-m}(y)dy \tag{4}$$
$$=\frac{(2x)^{n-\frac{1}{2}}\Gamma(n)}{2\sqrt{\pi}\Gamma(m)}\Big[2^{\frac{1}{2}m-n}\Gamma(\tfrac{1}{2}m-n)\,{}_1F_1(n\,;\,1+n-\tfrac{1}{2}m\,;\,\tfrac{1}{2}x^2)$$
$$+x^{m-2n}\frac{\Gamma(n-\frac{1}{2}m)\Gamma(\frac{1}{2}m)}{\Gamma(n)}\,{}_1F_1(\tfrac{1}{2}m\,;\,1-n+\tfrac{1}{2}m\,;\,\tfrac{1}{2}x^2)\Big]$$
$$R(m)>0,\ R(n)>0 \text{ and } R(n-m-1)<0$$

since, by the help of the known result*

$$D_n(x)=\frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}-\frac{1}{2}n)}\,2^{\frac{1}{2}n}e^{-\frac{1}{4}x^2}\,{}_1F_1(-\tfrac{1}{2}n\,;\,\tfrac{1}{2}\,;\,\tfrac{1}{2}x^2)$$
$$+\frac{\Gamma(-\frac{1}{2})}{\Gamma(-\frac{1}{2}n)}\,2^{\frac{1}{2}n-\frac{1}{2}}xe^{-\frac{1}{4}x^2}\,{}_1F_1(\tfrac{1}{2}-\tfrac{1}{2}n\,;\,\tfrac{3}{2}\,;\,\tfrac{1}{2}x^2)$$

it reduces, for $m=2\nu$ and $n=\nu+\frac{1}{2}$, to (2). The integral (3) is important from another point of view as well. It gives with the help of the integral (4) the following

THEOREM: *If*

$$f(x)=x^{\nu-\frac{1}{2}}e^{\frac{1}{4}x^2}D_{-2\nu-2}(x)$$

and

$$g(x)=\frac{x^{\nu+\frac{1}{2}}e^{\frac{1}{4}x^2}D_{-2\nu-1}(x)}{2\nu+1}$$

then $f(x)$ and $g(x)$ are J_ν-transforms of each other, provided $R(\nu)>-\frac{1}{2}$.

This Theorem is established in § 3. Finally in §§ 5-6, the values of the integrals

$$\int_0^\infty y^{2\nu+1}e^{\frac{1}{4}y^2\left(1-\frac{1}{2p^2}\right)}I_\nu\left(\frac{y^2}{8p^2}\right)D_{-m}(y)dy$$
$$\int_0^\infty y^{2\nu}e^{\frac{1}{4}y^2\left(1-\frac{1}{2p^2}\right)}I_\nu\left(\frac{y^2}{8p^2}\right)D_{-m}(y)dy$$

are given, by the help of (3) and (4) respectively, in terms of hypergeometric series.

2. Using Whittaker's integral† for $D_n(x)$, viz.,

$$D_n(x)=\frac{1}{\Gamma(-n)}e^{-\frac{1}{4}x^2}\int_0^\infty e^{-tx-\frac{1}{2}t^2}t^{-n-1}dt \qquad R(n)<0$$

we have

* Whittaker and Watson, *Modern Analysis* (4th Edition), p. 347.

† E. T. Whittaker, "On the functions associated with the parabolic cylinder in harmonic analysis", *Proc. Lond. Math. Soc.* (1), 35 (1903), 417-427.

$$\int_0^\infty y^{n+1}J_n(xy)e^{\frac{1}{4}y^2}D_{-m}(y)dy$$

$$=\frac{1}{\Gamma(m)}\int_0^\infty y^{n+1}J_n(xy)dy\int_0^\infty e^{-ty-\frac{1}{2}t^2}t^{m-1}dt \qquad R(m)>0$$

$$=\frac{1}{\Gamma(m)}\int_0^\infty e^{-\frac{1}{2}t^2}t^{m-1}dt\int_0^\infty e^{-ty}y^{n+1}J_n(xy)dy$$

$$=\frac{x^n\Gamma(2n+2)}{2^n\Gamma(m)\Gamma(n+1)}\int_0^\infty \frac{t^m e^{-\frac{1}{2}t^2}}{(t^2+x^2)^{n+\frac{3}{2}}}dt \qquad R(n+1)>0$$

since

$$\int_0^\infty e^{-ty}y^{n+1}J_n(xy)dy$$

$$=\frac{x^n}{2^n t^{2n+2}}\frac{\Gamma(2n+2)}{\Gamma(n+1)}\,{}_2F_1\left(n+1,\, n+\tfrac{3}{2}\,;\, n+1\,;-\frac{x^2}{t^2}\right)$$

$$=\frac{tx^n\Gamma(2n+2)}{2^n\Gamma(n+1)(t^2+x^2)^{n+\frac{3}{2}}}.$$

Using the known result

$$2\int_0^\infty \frac{x^{m-1}e^{-\frac{1}{2}x^2}}{(x^2+a^2)^n}dx=2^{\frac{1}{2}m-n}\Gamma(\tfrac{1}{2}m-n)\,{}_1F_1(n\,;\,1+n-\tfrac{1}{2}m\,;\,\tfrac{1}{2}a^2)$$

$$+a^{m-2n}\frac{\Gamma(n-\frac{1}{2}m)\Gamma(\frac{1}{2}m)}{\Gamma(n)}\,{}_1F_1(\tfrac{1}{2}m\,;\,1-n+\tfrac{1}{2}m\,;\,\tfrac{1}{2}a^2)$$

$$R(m)>0$$

we at once arrive at the integral (3).

If we put $n=\nu$ and $m=2\nu+3$ and make use of the relation (5), we obtain the integral equation (2) as a particular case of the integral (3).

To justify the change in the order of integration, suppose that

$$\theta(t)=t^{m-1}e^{-\frac{1}{2}t^2}\int_0^\infty e^{-ty}y^{n+1}J_n(xy)dy$$

$$=\frac{\Gamma(2n+2)x^n t^m e^{-\frac{1}{2}t^2}}{2^n\Gamma(n+1)(t^2+x^2)^{n+\frac{3}{2}}}$$

and
$$\phi(y)=y^{n+1}J_n(xy)\int_0^\infty e^{-ty-\frac{1}{2}t^2}t^{m-1}dt$$

For real values of x, $\theta(t)$ is uniformly convergent for the unlimited range $t\geqslant 0$ provided that $R(m-1)>0$ and $R(n+1)>0$. Since for all values of n,

$$J_n(x)=\sum_{r=0}^{\infty}\frac{(-1)^r x^{n+2r}}{2^{n+2r}r!\Gamma(n+r+1)}$$

$\phi(y)$ converges uniformly in $y\geqslant 0$, provided that $R(n+\frac{1}{2})>0$ and $R(m)>0$. Again consider the integral

$$I=\int_{\tau}^{\infty}|y^{n+1}J_n(xy)|dy\int_0^{\infty}e^{-ty-\frac{1}{2}t^2}t^{m-1}dt$$

where τ is large.

Now for large values of y,

$$J_n(y)=O(y^{-\frac{1}{2}})$$

and

$$e^{-ty}<(ty)^{-d-1},$$

where $R(d)>0$.

If we choose $d=n+1$, where $R(n+1)>0$, I is less than a constant multiple of

$$x^{-\frac{1}{2}}\int_{\tau}^{\infty}y^{-\frac{3}{2}}dy\int_0^{\infty}e^{-\frac{1}{2}t^2}t^{m-n-3}dt$$

which tends to zero when $R(m-n-2)>0$.

Hence the inversion is justified if $R(m-1)>0$, $R(n+\frac{1}{2})>0$ and $R(m-n-2)>0$. But by the theory of analytic continuation, the integral (3) is true for the more extensive ranges of m and n stated in § 1.

3. For $n=\nu$ and $m=2\nu+1$, (3) gives

$$\int_0^{\infty}\sqrt{xy}\,J_\nu(xy)\frac{y^{\nu+\frac{1}{2}}e^{\frac{1}{4}y^2}D_{-2\nu-1}(y)}{2\nu+1}dy$$

$$=x^{\nu-\frac{1}{2}}e^{\frac{1}{4}x^2}D_{-2\nu-2}(x) \qquad (6)$$

$R(\nu)>-\frac{1}{2}$.

and by putting $n=\nu+\frac{1}{2}$ and $m=2\nu+2$ in (4), we obtain

$$\int_0^{\infty}\sqrt{xy}\,J_\nu(xy)y^{\nu-\frac{1}{2}}e^{\frac{1}{4}y^2}D_{-2\nu-2}(y)dy$$

$$=\frac{x^{\nu+\frac{1}{2}}e^{\frac{1}{4}x^2}D_{-2\nu-1}(x)}{2\nu+1} \qquad R(\nu)>-\tfrac{1}{2}.$$

(7)

The integrals (6) and (7) at once lead us to the important Theorem enunciated in § 1.

4. We shall now give a few more interesting special cases of our general results (3) and (4).

Thus for $n=-\frac{1}{2}$ and $n=\frac{1}{2}$, the integral (3) reduces, by virtue of the known relations,

$$J_{-\frac{1}{2}}(z)=\left(\frac{2}{\pi z}\right)^{\frac{1}{2}}\cos z, \text{ and } J_{\frac{1}{2}}(z)=\left(\frac{2}{\pi z}\right)^{\frac{1}{2}}\sin z$$

to

$$\int_0^\infty \cos xy\, e^{\frac{1}{4}y^2} D_{-m}(y)\,dy=\frac{1}{2\Gamma(m)}\Big[\Gamma(\tfrac{1}{2}-\tfrac{1}{2}m)\Gamma(\tfrac{1}{2}+\tfrac{1}{2}m)x^{m-1}e^{\frac{1}{2}x^2}$$
$$+2^{\frac{1}{2}m-\frac{1}{2}}\Gamma(\tfrac{1}{2}m-\tfrac{1}{2})\,{}_1F_1(1\,;\tfrac{3}{2}-\tfrac{1}{2}m\,;\tfrac{1}{2}x^2)\Big],$$

valid when $R(m)>0$, odd integral values of $R(m)$ being excluded, and

$$\int_0^\infty y\sin xy\, e^{\frac{1}{4}y^2} D_{-m}(y)\,dy=\frac{x}{\Gamma(m)}\Big[2^{\frac{1}{2}m-\frac{3}{2}}\Gamma(\tfrac{1}{2}m-\tfrac{3}{2})$$
$$\times{}_1F_1(2\,;\tfrac{5}{2}-\tfrac{1}{2}m\,;\tfrac{1}{2}x^2)$$
$$+x^{m-3}\Gamma(\tfrac{3}{2}-\tfrac{1}{2}m)\Gamma(\tfrac{1}{2}+\tfrac{1}{2}m)\,{}_1F_1(\tfrac{1}{2}+\tfrac{1}{2}m\,;\tfrac{1}{2}m-\tfrac{1}{2}\,;\tfrac{1}{2}x^2)\Big],$$

true when $R(m)>1$, odd integral values of $R(m)$ being excluded.

The integral (4) for $n=1$ gives

$$\int_0^\infty y^{-1}e^{\frac{1}{4}y^2}\sin xy D_{-m}(y)\,dy=\frac{x}{2\Gamma(m)}\Big[\Gamma(\tfrac{1}{2}m)\Gamma(1-\tfrac{1}{2}m)x^{m-2}e^{\frac{1}{2}x^2}$$
$$+2^{\frac{1}{2}m-1}\Gamma(\tfrac{1}{2}m-1)\,{}_1F_1(1\,;2-\tfrac{1}{2}m\,;\tfrac{1}{2}x^2)\Big],$$

valid when $R(m)>0$, even integral values of $R(m)$ being excluded.

5. Writing 2ν for n in (3), we get

$$\int_0^\infty y^{2\nu+1}J_{2\nu}(xy)\,e^{\frac{1}{4}y^2}D_{-m}(y)\,dy \tag{8}$$
$$=\frac{x^{2\nu}\Gamma(4\nu+2)}{2^{2\nu+1}\Gamma(m)\Gamma(2\nu+1)}\Big[2^{\frac{1}{2}m-2\nu-1}\Gamma(\tfrac{1}{2}m-2\nu-1)$$
$${}_1F_1(2\nu+\tfrac{3}{2}\,;2+2\nu-\tfrac{1}{2}m\,;\tfrac{1}{2}x^2)$$
$$+x^{m-4\nu-2}\frac{\Gamma(1+2\nu-\frac{1}{2}m)\Gamma(\frac{1}{2}m+\frac{1}{2})}{\Gamma(2\nu+\frac{3}{2})}\,{}_1F_1(\tfrac{1}{2}m+\tfrac{1}{2}\,;-2\nu+\tfrac{1}{2}m\,;\tfrac{1}{2}x^2)\Big]$$

Multiply both sides of (8) by $e^{-p^2x^2}$ and integrate with respect to x between the limits 0 and ∞. The left hand side of (8) then becomes

$$\int_0^\infty e^{-p^2x^2}dx\int_0^\infty y^{2\nu+1}J_{2\nu}(xy)\,e^{\frac{1}{4}y^2}D_{-m}(y)\,dy$$

which, on changing the order of integration—a process obviously permissible—and then using the known integral*

$$\int_0^\infty J_{2\nu}(at)\exp(-p^2t^2)\,dt=\frac{\sqrt{\pi}}{2p}\exp\left(-\frac{a^2}{8p^2}\right)I_\nu\left(\frac{a^2}{8p^2}\right)$$

reduces to

$$\frac{\sqrt{\pi}}{2p}\int_0^\infty y^{2\nu+1}e^{\frac{1}{4}y^2\left(1-\frac{1}{2p^2}\right)}I_\nu\left(\frac{y^2}{8p^2}\right)D_{-m}(y)\,dy.$$

Now it is easy to prove that

$$\int_0^\infty x^l e^{-p^2x^2}{}_1F_1(a\,;\,b\,;\,\tfrac{1}{2}x^2)\,dx$$

$$=\frac{\Gamma(\frac{1}{2}l+\frac{1}{2})}{2p^{l+1}}\,{}_2F_1\left(a,\,\tfrac{1}{2}l+\tfrac{1}{2}\,;\,b\,;\,\frac{1}{2p^2}\right)$$

$$2p^2>1.$$

By the help of this the right side of (8) yields

$$\frac{\Gamma(4\nu+2)}{2^{2\nu+1}\Gamma(m)\Gamma(2\nu+1)}\Bigg[2^{\frac{1}{2}m-2\nu-1}\frac{\Gamma(\frac{1}{2}m-2\nu-1)\Gamma(\nu+\frac{1}{2})}{2p^{2\nu+1}}$$

$$\times{}_2F_1\left(2\nu+\tfrac{3}{2},\,\nu+\tfrac{1}{2}\,;\,2+2\nu-\tfrac{1}{2}m\,;\,\frac{1}{2p^2}\right)$$

$$+\frac{\Gamma(1+2\nu-\frac{1}{2}m)\Gamma(\frac{1}{2}m+\frac{1}{2})\Gamma(\frac{1}{2}m-\nu-\frac{1}{2})}{2p^{m-2\nu-1}\Gamma(2\nu+\frac{3}{2})}$$

$$\times{}_2F_1\left(\tfrac{1}{2}m+\tfrac{1}{2},\,\tfrac{1}{2}m-\nu-\tfrac{1}{2}\,;-2\nu+\tfrac{1}{2}m\,;\,\frac{1}{2p^2}\right)\Bigg].$$

It follows therefore that, when $R(m)>0$, $R(2\nu+1)>0$ and $R(2\nu-m+1)<0$

$$\int_0^\infty y^{2\nu+1}e^{\frac{1}{4}y^2\left(1-\frac{1}{2p^2}\right)}I_\nu\left(\frac{y^2}{8p^2}\right)D_{-m}(y)\,dy$$

$$=\frac{\Gamma(4\nu+2)}{\sqrt{\pi}\,2^{2\nu+1}\Gamma(m)\Gamma(2\nu+1)}\Bigg[2^{\frac{1}{2}m-2\nu-1}\frac{\Gamma(\frac{1}{2}m-2\nu-1)\Gamma(\nu+\frac{1}{2})}{p^{2\nu}}$$

$${}_2F_1\left(2\nu+\tfrac{3}{2},\,\nu+\tfrac{1}{2}\,;\,2+2\nu-\tfrac{1}{2}m\,;\,\frac{1}{2p^2}\right)$$

$$+\frac{\Gamma(1+2\nu-\frac{1}{2}m)\Gamma(\frac{1}{2}m+\frac{1}{2})\Gamma(\frac{1}{2}m-\nu-\frac{1}{2})}{p^{m-2\nu-2}\Gamma(2\nu+\frac{3}{2})}$$

$${}_2F_1\left(\tfrac{1}{2}m+\tfrac{1}{2},\,\tfrac{1}{2}m-\nu-\tfrac{1}{2}\,;-2\nu+\tfrac{1}{2}m\,;\,\frac{1}{2p^2}\right)\Bigg]$$

$$2p^2>1.$$

* Watson, *Bessel Functions*, p. 394.

6. Writing the integral (4) in the form

$$\int_0^\infty y^{2\nu} e^{\frac{1}{4}y^2} J_{2\nu}(xy) D_{-m}(y)\,dy$$

$$= \frac{(2x)^{2\nu}\Gamma(2\nu+\frac{1}{2})}{2\sqrt{\pi}\,\Gamma(m)}\Big[2^{\frac{1}{2}m-2\nu-\frac{1}{2}}\Gamma(\tfrac{1}{2}m-2\nu-\tfrac{1}{2})$$

$$\times {}_1F_1(2\nu+\tfrac{1}{2};\tfrac{3}{2}+2\nu-\tfrac{1}{2}m;\tfrac{1}{2}x^2)$$

$$+x^{m-4\nu-1}\frac{\Gamma(2\nu+\frac{1}{2}-\frac{1}{2}m)\Gamma(\frac{1}{2}m)}{\Gamma(2\nu+\frac{1}{2})}\,{}_1F_1(\tfrac{1}{2}m;\tfrac{1}{2}-2\nu+\tfrac{1}{2}m;\tfrac{1}{2}x^2)\Big]$$

and proceeding with it in the manner of § 5, we obtain

$$\int_0^\infty y^{2\nu} e^{\frac{1}{4}y^2\left(1-\frac{1}{2p^2}\right)} I_\nu\left(\frac{y^2}{8p^2}\right) D_{-m}(y)\,dy$$

$$2p^2>1.$$

$$= \frac{2^{2\nu-1}\Gamma(2\nu+\frac{1}{2})}{\pi\Gamma(m)}\Big[2^{\frac{1}{2}m-2\nu-\frac{1}{2}}\frac{\Gamma(\frac{1}{2}m-2\nu-\frac{1}{2})\Gamma(\nu+\frac{1}{2})}{p^{2\nu}}$$

$$\times {}_2F_1\left(2\nu+\tfrac{1}{2},\nu+\tfrac{1}{2};\tfrac{3}{2}+2\nu-\tfrac{1}{2}m;\frac{1}{2p^2}\right)$$

$$+\frac{\Gamma(2\nu+\frac{1}{2}-\frac{1}{2}m)\Gamma(\frac{1}{2}m)\Gamma(\frac{1}{2}m-\nu)}{p^{m-2\nu-1}\Gamma(2\nu+\frac{1}{2})}$$

$$\times {}_2F_1\left(\tfrac{1}{2}m,\tfrac{1}{2}m-\nu;\tfrac{1}{2}+\tfrac{1}{2}m-2\nu;\frac{1}{2p^2}\right)\Big],$$

valid when $R(m)>0$, $R(2\nu+\frac{1}{2})>0$ and $R(2\nu-m)<0$.

"ON SOME k_n-FUNCTION FORMULAE"

By N. G. SHABDE, College of Science, Nagpur

[Received 23 June 1937]

1. INTRODUCTION:—The object of this paper is to collect a number of results involving k_n-functions. These formulae do not seem to have been noted explicitly as yet. Some of them are obtained by the methods of operational calculus and others are derived from formulae involving $W_{k,m}$ or $M_{k,m}$ functions and Laguerre or Sonine polynomials given by Erdelyi,* Howell,† Bailey‡ and Bateman.§ The k_n-functions are related to other functions by the relations

$$W_{\nu,\frac{1}{2}}(t)=\Gamma(\nu+1)k_{2\nu}(t/2) \tag{1}$$

$$M_{n,\frac{1}{2}}(t)=(-1)^{n-1}k_{2n}(t/2) \tag{2}$$

$$k_{2n}(x)=(-1)^{n-1}(2x).\frac{e^{-x}}{n}.L_{n-1}^{(1)}(2x) \tag{3}$$

$$T_1^{n-1}(2s)=\frac{e^s k_{2n}(s)}{(2s)(n-1)!}. \tag{4}$$

The Theorems in the operational calculus, which are made use of in this paper, are given by Nissen K. F.‖ Only the results are given in this paper and lengthy algebra has been suppressed.

* (i) "Funktionalrelationen mit konfluenthypergeometrischen Funktionen", *Math. Zeit.*, 42, 125-43.

(ii) "Über eine Methode zur Gewinnung von Funktionalbezeihungen zwischen konfluenten hypergeometrischen Funktionen", *Monatshefte für Mathematik und Physik*, 45, 31-52.

(iii) Über die erzeugende Funktion der Jacobischen Polynome, *Math. Zeit.*. 40, 693-702.

† "A note on Laguerre Polynomials", *Phil. Mag.*, (7) 23 (1937), Supp. number.

‡ "On the product of two Legendre Polynomials with different arguments", *Proc. Lond. Math. Soc.* (2), 41, 215-220.

§ The Partial Differential Equations in Mathematical Physics, pp. 451-459. Bateman has collected in this book results involving Sonine Polynomials due to Wilson, Koshliakov and other authors.

‖ "A contribution to the Symbolic Calculus", *Phil. Mag.* (7) 20 (1935), 977-997.

2. THE FORMULAE:—

(1) $$(-1)^{m+n-2} m!\, n!\, k_{2n}(x) k_{2m}(x)$$
$$=4x^2 \sum_{s=0}^{\infty} \frac{\Gamma(n+s+1)\Gamma(m+s+1)\Gamma(m+n+1)}{s!\,(s+2)\Gamma(m+n+2+2s)} (2x)^{2s+1} \times L_{n+m}^{2s+1}(2x)$$

(2) $$k_{2m}(x) k_{2n}(x)$$
$$=(-1)^{m+n-2} 2x \frac{1}{m!\,n!} \sum_{s=0}^{\infty} (-1)^{2s+1} \frac{\Gamma(n+s+1)\Gamma(m+s+1)}{s!\,\Gamma(s+2)} \times L_{m+n+2s+1}^{-1-2s}(2x)$$

(3) $$\sum_{n=1}^{\infty} n t^{n-1} k_{2n}(x) k_{2n}(y)$$
$$=2\sqrt{xy}\frac{t^{-\frac{1}{2}}}{1-t}\exp\left\{-(x+y)\cdot\frac{1+t}{1-t}\right\} I_1\left\{\frac{4\sqrt{xyt}}{1-t}\right\};\ |t|<1$$

(4) $$\int_0^{\infty} \frac{e^{-t} J_1(2\sqrt{2xt})\, k_{2n}(t)\, dt}{\sqrt{2t}} = \frac{(-1)^{n-1} x^{n-\frac{1}{2}} e^{-x}}{2(n!)}$$

(5) $$\int_0^{\infty} e^{-t}(2t)^{\frac{n}{2}-1} J_{2-n}(4\sqrt{xt})\, k_{2n}(t)\, dt = \frac{e^{-x}}{2}(2x)^{\frac{n}{2}-1} k_{2n}(x)$$

(6) $$n\rho \int_0^{\infty} \frac{e^{\lambda^2\tau}\cdot e^{-\frac{\rho^2}{8\tau}}\cdot \tau^{-n} k_{2n}\left(\frac{\rho^2}{8\tau}\right) d\tau}{\rho^2} = -\int_0^{\infty} \frac{v^{2n} J_1(\rho v)\, dv}{v^2+\lambda^2}$$

(7) $$k_{2n}(x+y) = \frac{2(-1)^{n-1}(x+y)e^{y-x}}{n} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} (2y)^m L_{n-1}^{1+m}(2x)$$

(8) $$n\int_0^x e^{\frac{(\xi+x)}{2}} \cdot k_{2n}\left(\frac{x-\xi}{2}\right) d\xi = (-1)^{n-1} x e^x \int_0^{\infty} e^{-t}\cdot t^{n-1}\cdot I_2(2\sqrt{tx})\, dt$$

(9) $$k_{2n}(x) = (n-1)!\ e^{-x/2} \sum_{m=0}^{n-1} \frac{2^{n-m}}{m!\,(n-1-m)!} k_{2n-2m}(x/2)$$

(10) $$\int_0^1 e^{xy} k_{2n}(xy)(1-y)^{\rho-1} dy$$
$$= \frac{(n-1)!\,\Gamma(\rho)(-1)^{n-1}}{\Gamma(\rho+n+1)} L_{n-1}^{(\rho+1)}(2x)\ ;\ \rho>0$$

$$(11)\quad \int_0^s e^{s/2} k_{2n+2}\left(\frac{s-t}{2}\right) k_{2\nu+2}(t/2)\,dt$$

$$=\Gamma(n+\nu+1)(-1)^{n+\nu}\frac{s^3 L^3_{n+\nu}(s)}{\Gamma(4+n+s)}$$

$$(12)\quad \int_0^\infty \frac{e^{\frac{-s(2x-a-b)}{2}} k_{2n+2}\left(\frac{as}{2}\right) k_{2\nu+2}\left(\frac{bs}{2}\right) ds}{s}$$

$$=(-1)^{n+\nu}\frac{\Gamma(n+\nu+2)ab(x-a)^n(x-b)^\nu}{\Gamma(n+2)\Gamma(\nu+2)x^{n+\nu+2}}$$

$$\times F\left(-n,-\nu;-1-n-\nu;\frac{x(x-a-b)}{(x-a)(x-b)}\right);\ a>0,\ b>0$$

$$(13)\quad \int_0^\pi U_{2n}(\sqrt{s}\cos\psi)\sin^2\psi\,d\psi=\frac{\pi}{2}.\frac{(2n)!}{n!}\frac{e^{s/2}}{s}k_{2n+2}(s/2)$$

where
$$U_n(x)=(-1)^n e^{x^2}\frac{d^n}{dx^n}(e^{-x^2})$$

$$(14)\quad \frac{2(p-1)^{n-1}}{p^{m-2}(1+p)^{n+1}}\doteqdot\int_0^\infty\left(\frac{x}{s}\right)^{\frac{m}{2}}J_m(2\sqrt{xs})k_{2n}(s)\,ds.$$

When $m=1$, this gives

$$(-1)^{n-1}k_{2n}(x)=\int_0^\infty\left(\frac{x}{s}\right)^{\frac{1}{2}}J_1(2\sqrt{xs})k_{2n}(s)\,ds$$

$$(15)\quad \frac{2p\left\{1-\sqrt{1+p^2}\right\}^{n-1}}{\left\{1+\sqrt{1+p^2}\right\}^{n+1}}\doteqdot k_{2n}(x)-\int_0^x k_{2n}(\xi)\frac{\xi J_1(\sqrt{x^2-\xi^2})\,d\xi}{\sqrt{x^2-\xi^2}}$$

$$\doteqdot k_{2n}(x)-\int_0^x k_{2n}\left\{\sqrt{x^2-s^2}\right\}J_1(s)\,ds$$

$$(16)\quad \frac{2p^2(p-p^2-1)^{n-1}}{(p+1+p^2)^{n+1}}\doteqdot\int_0^x J_0\left\{2\sqrt{s(x-s)}\right\}k_{2n}(s)\,ds$$

$$(17)\quad \int_0^x(\xi-x)^m k_{2n}(\xi)\,d\xi$$

$$=\Pi(m)(-1)^{n-1}\int_0^\infty\left(\frac{x}{s}\right)^{\frac{m}{2}+1}J_{m+2}(2\sqrt{xs})k_{2n}(s)\,ds;\ m>0$$

$$(18)\quad e^{-s/2}\frac{s^2}{n}.\frac{d}{ds}\left[\frac{e^{s/2}.k_{2n+2}(s/2)}{s}\right]=k_{2n}(s/2)+k_{2n+2}(s/2)$$

$$(19)\quad \frac{d^p}{ds^p}\left[\frac{e^{s/2}.k_{2n+2}(s/2)}{s.n!}\right]=T^{n-p}_{1+p}(s)$$

$$(20)\quad \frac{d^p}{ds^p}\left[\frac{e^{s/2}k_{2n+2}(s/2)}{n!}\right]=s^{1-p}(-1)^n\frac{L_n^{1-p}(s)}{\Gamma(2-p+n)}$$

$$(21)\quad s^2T_2^n(s)=\frac{e^{s/2}}{n!}k_{2n+4}(s/2)+\frac{e^{s/2}}{n!}k_{2n+2}(s/2)$$

$$(22)\quad \frac{e^{s/2}k_{2n}(s/2)}{s(n-1)!}=(n+1)T_2^{n-1}(s)+T_2^{n-2}(s)$$

$$(23)\quad \frac{k_{2n}(s/2)e^{s/2}}{(n-2)!}=(s-n)T_2^{n-2}(s)-T_2^{n-3}(s)$$
$$=(s-2)T_2^{n-2}(s)-sT_3^{n-3}(s)$$

$$(24)\quad (n+2)\frac{d}{ds}\left[\frac{e^{s/2}k_{2n+4}(s/2)}{(n+1)!\,s}\right]$$
$$=\frac{e^{s/2}k_{2n+2}(s)}{n!\,s}-\frac{d}{ds}\left[\frac{e^{s/2}.k_{2n+2}(s/2)}{s.n!}\right]$$

$$(25)\quad \frac{d}{ds}\left[\frac{s^{-1}e^sk_{2n+4}(s/2)}{(n+1)k_{2n+2}(s/2)}\right]=e^ss^{-2}\left[\frac{e^{s/2}k_{2n+2}(s/2)}{n!\,s}\right]^{-2}$$
$$\times\left[(n+1)\left\{\frac{e^{s/2}k_{2n+4}(s/2)}{(n+1)!\,s}\right\}^2\right.$$
$$\left.+\left\{\frac{e^{s/2}k_{2n+2}(s/2)}{n!\,s}+(n+1)\frac{e^{s/2}k_{2n+4}(s/2)}{n!\,s}\right\}^2\right].$$

TRAJECTORIES AND LINES OF FORCE IN A RIEMANNIAN SPACE*

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[Received 5 July 1937]

Kasner† has proved the following Theorems relating to the curvatures of the Trajectories and Lines of force in a plane :—

Theorem I. The curvature of the Trajectory obtained by starting a particle from rest in any field of force is one-third the curvature of the line of force through the given point.

Theorem II. If the line of force has contact of nth order with the tangent line, the trajectory produced by starting a particle from rest will also have contact of the nth order and the limiting ratio of the departure of the trajectory to the departure of the line of force from the common tangent will be $1:2n+1$.

Theorem III. If a particle is projected in the direction of the force with a speed different from zero, the initial curvature will be zero and the infinitesimal departure from the common tangent will vary inversely as the square of the speed. That is $d\gamma/ds$ varies as $1/v^2$.

Theorem IV. The single infinity of paths obtained by starting at a given point in the force direction with varying speeds under the conditions of Theorem II will have contact of order $(n+1)$ with the common tangent, and will give departures from the common tangent varying inversely as the square of the speed; except for the single path due to zero speed for which case the contact will be of the nth order and the departure ratio will be of the form $1:2n+1$.

Theorem V. If R_0, the resistance due to zero speed at a point does not vanish, as in the case of sliding friction, the ratio

* I express my grateful thanks to Professor A. Narasinga Rao who suggested this problem and under whose guidance this investigation was carr'ed out.

† Edward Kasner. 'General Theorems on Trajectories and Lines of Force', *Proceedings of the National Academy of Sciences U.S.A.*, 20 (1934), 130-135.

of the initial curvatures of the trajectory and the line of force for a particle starting from rest is $1 : 3+2R_0/F$, where F is the acting force at the position where the particle starts from rest.

The object of this paper is to extend the above results to a Riemannian space of n-dimensions.

1. Let $(x^1, x^2, \ldots x^n)$ be the co-ordinates of a point in a Riemannian n-space. Then $(\dot{x}^1, \dot{x}^2, \ldots \dot{x}^n)$ where dots denote differentiation with respect to the time t are the generalised velocities and $\delta\dot{x}^i/\delta t$ $(i=1, 2, \ldots n)$, where $\delta/\delta t$ is the covariant time flux operator, are the accelerations. Then if f^i denote the component of the accelerations, we have

$$f^i = \frac{\delta \dot{x}^i}{\delta t} = \ddot{x}^i + \Gamma^i_{jl}\dot{x}^j\dot{x}^l$$

$$= \ddot{s}\lambda^i_0 + \kappa_1 \dot{s}^2 \lambda^i_1,^{*} \tag{1·1}$$

where λ^i_0, λ^i_1 are the components of the unit tangent and the first normal to the trajectory and κ_1 is its first curvature.

2. We now prove the following:

LEMMA. *If C and $\bar{C}$ be two curves having contact of order p at a point O, then*

$$\left(\frac{d^n}{ds^n}\right)_0 = \left(\frac{d^n}{d\sigma^n}\right)_0 \qquad (n=1, 2, \ldots p),$$

where s and σ are the arc lengths measured along C and $\bar{C}$ respectively and the suffix 0, *denotes the value at the point O.*

Let P and Q be two points at equal infinitesimal arc lengths s from O measured along the curves. Then C and $\bar{C}$ are said to have contact of order p if PQ is an infinitesimal of order s^{p+1}.†

Then, if ϕ is any function of position, we have

$$\frac{d\phi}{ds} = \frac{d\phi}{d\sigma}\frac{d\sigma}{ds} + \frac{d\phi}{d\eta}\frac{d\eta}{ds}, \quad \text{where } \eta = PQ.$$

When C and $\bar{C}$ touch at O, $\dfrac{d\eta}{ds} \rightarrow$ zero and since $s=\sigma$ we have $\dfrac{d\sigma}{ds} = 1$ always. Hence

*J. L. Synge 'On the Geometry of Dynamics', *Phil. Trans. Royal Society of London*, A 226 (1926).

†A. J. McConnell, 'The contact of curves in a Riemannian space', *Proc. Lond. Math. Soc.* (1927), 512.

$$\left(\frac{d}{ds}\right)_0=\left(\frac{d}{d\sigma}\right)_0 \tag{2·1}$$

Putting $\frac{d}{ds}$ in the form $\frac{d}{ds}=\frac{d\sigma}{ds}A+\frac{d\eta}{ds}B$, where $A=\frac{d}{d\sigma}$ and $B=\frac{d}{d\eta}$ we have, as a result of successive differentiations,

$$\frac{d^n}{ds^n}=\sum_{r=0}^{n-1}\binom{n-1}{r}\frac{d^{r+1}\sigma}{ds^{r+1}}A^{(n-r-1)}+\sum_{r=0}^{n-1}\binom{n-1}{r}\frac{d^{r+1}\eta}{ds^{r+1}}B^{(n-r-1)}, \tag{2·2}$$

where $A^{(r)}$, $B^{(r)}$ stand for the rth derivative of the expressions with respect to s.

Since $\sigma=s=$the length measured along both the curves we have

$$\frac{d^r\sigma}{ds^r}=0 \quad (r=2, 3, \ldots n) \tag{2·3}$$

Also, when C and $\bar{C}$ have contact of order p, η is an infinitesimal of order $s^{(p+1)}$ and hence $d^r\eta/ds^r$ is an infinitesimal of order s^{p+1-r} and vanishes if $r<p+1$

i.e. we have $\qquad d^r\eta/ds^r \to 0 \quad (r=1, 2, \ldots p) \qquad$ (2·4)

Then

$$\left(\frac{d^n}{ds^n}\right)_0=A_0^{(n-1)}. \quad (n=1, 2, \ldots p) \tag{2·5}$$

the other terms vanishing as a result of (2·3) and (2·4).

Also

$$A^{(n-1)}=\frac{d\sigma}{ds}\frac{d^n}{d\sigma^n}+\sum_{r=2}^{n-1}\frac{d^r\sigma}{ds^r}C_r+\sum_{r=1}^{n-1}\frac{d^r\eta}{ds^r}D_r$$

so that

$$A_0^{(n-1)}=\left(\frac{d^n}{d\sigma^n}\right)_0 \quad (n=1, 2, \ldots p) \tag{2·6}$$

From (2·5) and (2·6) we have the Lemma.

3. McConnell* has shown that the necessary and sufficient conditions for two curves to have contact of order p, are

$$\lambda_0^{i(n)} \quad (n=0, 1, \ldots p-1)$$

where $\lambda_0^{i(n)}=\frac{\delta^n\lambda_0^i}{\delta s^n}$, should be the same for both the curves. $\qquad$ (3·1)

* Loc. cit. p. 513.

Now

$$\lambda_0^{i(n)} = \sum_{r=0}^{n-1} \binom{n-1}{r} \kappa_1^{(r)} \lambda_1^{i(n-r-1)} \tag{3·2}$$

So, if $\kappa_1, \kappa'_1 \ldots \kappa_1^{(m-2)}$ are all zero for both the curves, we have $\lambda_0^{i(n)}=0$ $(n=0, 1, 2,\ldots,\ldots m-1)$ for both the curves by (3·2) and hence by (3·1) they have contact of order m. (3·3)

We shall make use of this result (3·3) in all the stages of our investigation. It must be noted that these are merely sufficient conditions.

4. Let C be the trajectory and $\bar{C}$ the line of force through the point O. Let f^r denote the contravariant components of the acceleration vector and Q^r those of the force vector. Also let $\lambda_\mu^r, \bar{\lambda}_\mu^r$ $(\mu=0, 1,\ldots n-1)$ denote the unit tangent and the $(n-1)$ normals to C and $\bar{C}$ respectively. Then

$$f^r = \ddot{s}\lambda_0^r + \kappa_1 \dot{s}^2 \lambda_1^r = Q^r = f\bar{\lambda}_0^r \tag{4.1}$$

where f is the magnitude of the force vector. If the tangential and normal components be denoted by T and N respectively, we have

$$f^r = T\lambda_0^r + N\lambda_1^r = f\bar{\lambda}_0^r.$$

Case I. Suppose a particle starts from rest at the point O.

Then $$N_0 = (\kappa_1 \dot{s}^2)_0 = 0$$

Hence $$(\lambda_0^r)_0 = (\bar{\lambda}_0^r)_0$$

and $$T_0 = f_0 \tag{4·2}$$

It follows that the trajectory and the line of force touch at O. Hence we have,

$$\left(\frac{\delta f^r}{\delta s}\right)_0 = \left(\frac{\delta f^r}{\delta \sigma}\right)_0 \text{ by the Lemma}$$

$$= \left(\frac{\delta Q^r}{\delta \sigma}\right)_0 \text{ since } f^r = Q^r \text{ always.} \tag{4·3}$$

Now

$$\begin{aligned} \frac{\delta f^r}{\delta s} &= (T' - \kappa_1 N)\lambda_0^r + (\kappa_1 T + N')\lambda_1^r + \kappa_2 N \lambda_2^r \\ \frac{\delta Q^r}{\delta \sigma} &= f'\bar{\lambda}_0^r + f\bar{\kappa}_1 \bar{\lambda}_1^r \end{aligned} \tag{4·4}$$

$N_0=0$ and $N'=(\kappa_1 v^2)'=\kappa'_1 v^2+2\kappa_1 T$, and so $N'_0=2\kappa_1 f_0$. Hence we have

$$\left.\begin{aligned} \left(\frac{\delta f^r}{\delta s}\right)_0 &= T'_0\lambda_0^r+3\kappa_1 f_0\lambda_1^r \\ \left(\frac{\delta Q^r}{\delta\sigma}\right)_0 &= f'_0\bar{\lambda}_0^r+f_0\bar{\kappa}_1\bar{\lambda}_1^r \end{aligned}\right\} \tag{4·5}$$

Now $\left(\frac{\delta f^r}{\delta s}\right)_0$ and $\left(\frac{\delta Q^r}{\delta\sigma}\right)_0$ are identical vectors by (4·3) and by the former of the equations (4·5) they lie in the osculating plane of the trajectory. Also since $(\lambda_0^r)_0=(\bar{\lambda}_0^r)_0$ by (4·2) and $\lambda_1^r, \bar{\lambda}_1^r$ are vectors perpendicular to them and lying in the same plane, we have

$$(\lambda_1^r)_0=(\bar{\lambda}_1^r)_0 \tag{4·6}$$

Hence equating the co-efficients we have

$$3\kappa_1 f_0=\bar{\kappa}_1 f_0 \text{ i.e. } \kappa_1=\bar{\kappa}_1/3 \text{ if } f_0\neq\text{zero.}$$

So we get

THEOREM 1. *The first curvature of a trajectory obtained when a particle starts from rest in any non-vanishing field of force is one-third the first curvature of the line of force through that point.*

5. When $\bar{\kappa}_1=0$, we have $\kappa_1=0$. Then by (3·3), the trajectory and the line of force have contact of order 2. So $f_0^{r(2)}=Q_0^{r(2)}$

$$\left.\begin{aligned} f_0^{r(2)} &= T''_0\lambda_0^r+5\kappa'_1 f_0\lambda_1^r \\ Q_0^{r(2)} &= f''_0\bar{\lambda}_0^r+\bar{\kappa}'_1 f_0\bar{\lambda}_1^r \end{aligned}\right\} \tag{5·1}$$

Equating the co-efficients of λ_1^r and $\bar{\lambda}_1^r$ we have

$$\kappa'_1=\bar{\kappa}'_1/5 \text{ when } f_0\neq\text{zero.} \tag{5·2}$$

Generalising from this, we have the following

THEOREM 2. *If a particle starts from rest at a point O and if $\bar{\kappa}_1$ and its first $(p-2)$ derivatives be all zero at O, then $\kappa_1, \kappa'_1 \ldots \kappa_1^{(p-2)}$ are also zero, so that the trajectory and line of force have contact of order p. Then $\kappa_1^{(p-1)}=\bar{\kappa}_1^{(p-1)}/(2p+1)$*

We shall prove this by the method of induction.

Assuming the theorem to be true for values upto $(p-1)$ we shall prove it holds good for p. We have $\bar{\kappa}_1, \bar{\kappa}'_1 \ldots \bar{\kappa}^{(p-3)}$ are all zero and $\kappa_1, \kappa'_1 \ldots \kappa_1^{(p-3)}$ are also zero. Hence they have contact of order $(p-1)$ and $\kappa_1^{(p-2)}=\bar{\kappa}_1^{(p-2)}/(2p-1)$. Now if $\bar{\kappa}_1^{(p-2)}=0$

we have $\kappa_1^{(p-2)}$ is also zero so that $\bar{\kappa}_1, \bar{\kappa}'_1 \ldots \bar{\kappa}_1^{(p-2)}$ and $\kappa_1, \kappa'_1 \ldots \kappa_1^{(p-2)}$ are all zero. Then by (3·3), the trajectory and the line of force have contact of order p. Hence $f_0^{r(p)} = Q_0^{r(p)}$.

Also

$$\begin{aligned} f^{r(p)} = {} & \lambda_0^r[T^{(p)} + A_m\kappa_1^{(m)}] + \lambda_1^r[B_m\kappa_1^{(m)} + (2p+1)\kappa_1^{(p-1)}T + \kappa_1^{(p)}v^2] \\ & + \lambda_2^r[C_m\kappa_1^{(m)} + p\kappa_2\kappa_1^{(p-1)}v^2] + \lambda_s^r[D_m^s\kappa_1^{(m)}] \end{aligned} \tag{5·3}$$

$$(m=0, 1, \ldots p-2), \ (s=3, 4, \ldots p+1), \ D_m^n = 0, \ m>1$$
$$\text{and } D_m^{n+1} = 0, \ m>0$$

Also

$$\begin{aligned} Q^{r(p)} = {} & \bar{\lambda}_0^r[f^{(p)} + \bar{A}_m\bar{\kappa}_1^{(m)}] + \bar{\lambda}_1^r[\bar{B}_m\bar{\kappa}_1^{(m)} + f\bar{\kappa}_1^{(p-1)}] \\ & + \bar{\lambda}_s^r[\bar{C}_m^s\bar{\kappa}_1^{(m)}] \end{aligned} \tag{5·4}$$

$$(m=0, 1 \ldots p-2), \ (s=2, 3, \ldots p), \ \bar{C}_m^{n-1} = 0; \ m>1$$
$$\text{and } \bar{C}_m^n = 0, \ m>0$$

Since $\kappa_1^{(m)} = 0 = \bar{\kappa}_1^{(m)} \ (m=0, 1, \ldots, p-2)$

we have

$$\left.\begin{aligned} f_0^{r(p)} &= \lambda_0^r[T^{(p)}] + \lambda_1^r[(2p+1)\kappa_1^{(p-1)}T] \\ Q_0^{r(p)} &= \bar{\lambda}_0^r[f_0^{(p)}] + \bar{\lambda}_1^r[f_0\bar{\kappa}_1^{(p-1)}] \end{aligned}\right\} \tag{5·5}$$

Equating the co-efficients of λ_1^r and $\bar{\lambda}_1^r$ we get

$$\kappa_1^{(p-1)} = \bar{\kappa}_1^{(p-1)}/(2p+1), \ f_0 \neq \text{zero}.$$

Hence the Theorem.

6. We have till now been discussing the motion of a particle that starts from rest at the point O. Now let us consider

Case II. The motion of a particle that is projected in the force-direction with a non-zero velocity v.

Since the infinitesimal displacement is in the direction of the force $N_0 = (\kappa_1 v^2)_0 = 0$ and hence $(\kappa_1)_0 = 0$ (6·1)
Also the line of force and the trajectory touch at O and hence $f_0^{r(1)} = Q_0^{r(1)}$,

$$\left.\begin{aligned} f_0^{r(1)} &= T'_0\lambda_0^r + \kappa'_1 v^2\lambda_1^r \\ Q_0^{r(1)} &= f'_0\bar{\lambda}_0^r + \bar{\kappa}_1 f_0\bar{\lambda}_1^r \end{aligned}\right\}. \tag{6·2}$$

Equating the coefficients of λ_1^r and $\bar{\lambda}_1^r$ we have

$$\kappa'_1 = f_0\bar{\kappa}_1/v^2 = A/v^2 \tag{6·3}$$

where A is a constant, since f_0 and $\bar{\kappa}_1$ have fixed non-zero values at O. Hence we have

THEOREM 3. *When a particle is projected in the direction of the line of force with a non-zero velocity v, its first curvature vanishes and the first derivative of the first curvature varies inversely as the square of the velocity.*

7. If $\bar{\kappa}_1=0$ we have $\kappa'_1=0$. Then since $\kappa_1=0$ and $\bar{\kappa}_1=0$ the trajectory and the line of force have contact of order 2, (by 3·3) and hence

$$f_0^{r(2)}=Q_0^{r(2)}$$

$$\left.\begin{aligned} f_0^{r(2)}&=T''_0\lambda_0^r+\kappa_1''v^2\lambda_1^r \\ Q_0^{r(2)}&=f''_0\bar{\lambda}_0^r+\bar{\kappa}'_1f_0\bar{\lambda}_1^r \end{aligned}\right\} \tag{7·1}$$

We get

$$\kappa''_1=f_0\bar{\kappa}'_1/v^2=A/v^2 \tag{7·2}$$

$$\text{if } \bar{\kappa}'_1 \neq \text{zero}.$$

Hence if $\bar{\kappa}'_1\neq$zero while $\bar{\kappa}=$zero, we get $\kappa_1=0$, $\kappa'_1=0$. The trajectory and the line of force have contact of order 2 and κ''_1 varies inversely as the square of the velocity. Generalising this we have

THEOREM 4. *If $\bar{\kappa}_1^{(p-1)}$ is the first derivative of $\bar{\kappa}_1$ that does not vanish, then $\kappa_1, \kappa'_1 \ldots \kappa_1^{(p-1)}$ all vanish. The trajectory and the line of force have contact of order p and $\kappa_1^{(p)}=A/v^2$ where $A=f_0\bar{\kappa}_1^{(p-1)}$.*

Let us as before prove this by induction. Suppose the Theorem to be true for values up to $(p-1)$. Then $\bar{\kappa}_1, \bar{\kappa}'_1 \ldots \bar{\kappa}_1^{(p-3)}$ are all zero and $\kappa_1, \kappa'_1 \ldots \kappa_1^{(p-2)}$ all vanish. The trajectory and the line of force have contact of order $(p-1)$ and $\kappa_1^{(p-1)}=A/v^2$ where $A=f_0\bar{\kappa}_1^{(p-2)}$.

Now suppose $\bar{\kappa}_1^{(p-2)}=0$. Then $\kappa_1^{(p-1)}=0$. Since $\bar{\kappa}_1, \bar{\kappa}'_1 \ldots \bar{\kappa}_1^{(p-2)}$ and $\kappa_1, \kappa'_1 \ldots \kappa_1^{(p-2)}$ all vanish, the trajectory and the line of force have contact of order p (by 3·3). So $f_0^{r(p)}=Q_0^{r(p)}$. From (5·3) and (5·4) we have, making use of the above conditions

$$\left.\begin{aligned} f_0^{r(p)}&=T^{(p)}\lambda_0^r+[\kappa_1^{(p)}v^2]\lambda_1^r \\ Q_0^{r(p)}&=f^{(p)}\bar{\lambda}_0^r+[f_0\bar{\kappa}_1^{(p-1)}]\bar{\lambda}_1^r \end{aligned}\right\} \tag{7·3}$$

Equating the coefficients of λ_1^r and $\bar{\lambda}_1^r$ we get

$$\kappa_1^{(p)}=f_0\bar{\kappa}_1^{(p-1)}/v^2=A/v^2 \tag{7·4}$$

Hence the Theorem.

8. *Motion in a Resisting Medium.* The equations of motion in a resisting medium are

$$f^r = \ddot{s}\lambda_0^r + \kappa_1 \dot{s}^2 \lambda_1^r = Q^r + R\lambda_0^r \tag{8·1}$$

i.e. $$(\ddot{s} - R)\lambda_0^r + \kappa_1 \dot{s}^2 \lambda_1^r = f\bar{\lambda}_0^r.$$

Denoting the left hand member as $\bar{f}^r$ we have

$$\bar{f}^r = Q^r \tag{8·2}$$

Denoting $\ddot{s} - R = T$ and $\kappa_1 \dot{s}^2 = N$

$$\left.\begin{aligned} \bar{f}^r &= T\lambda_0^r + N\lambda_1^r \\ Q^r &= f\bar{\lambda}_0^r \end{aligned}\right\} \tag{8·3}$$

CASE I. THE PARTICLE STARTS FROM REST AT THE POINT O.

$N_0 = \kappa_1 \dot{s}^2 = 0$ and hence $T_0 = f_0$ and $\lambda_0^r = \bar{\lambda}_0^r$. The trajectory and the line of force touch each other at O. Hence $\bar{f}_0^{r(1)} = Q_0^{r(1)}$.

$$\bar{f}^{r(1)} = (T' - \kappa_1 N)\lambda_0^r + (\kappa_1 T + N')\lambda_1^r + \kappa_2 N \lambda_2^r$$

$$N' = \kappa'_1 v^2 + \kappa_1 (v^2)' = \kappa'_1 v^2 + 2\kappa_1 \ddot{s} = \kappa'_1 v^2 + 2\kappa_1 (T + R).$$

Hence

$$\left.\begin{aligned} \bar{f}_0^{r(1)} &= T'\lambda_0^r + \kappa_1 (3T + 2R)\lambda_1^r \\ Q_0^{r(1)} &= f'\bar{\lambda}_0^r + f_0 \bar{\kappa}_1 \bar{\lambda}_1^r \end{aligned}\right\} \tag{8·4}$$

Equating the coefficients of λ_1^r and $\bar{\lambda}_1^r$ we get

$$\kappa_1 (3T_0 + 2R_0) = \bar{\kappa}_1 f_0$$

i.e. $$\kappa_1 : \bar{\kappa}_1 = 1 : 3 + \frac{2R_0}{f_0} \tag{8·5}$$

Hence we have the analogue of Theorem I for a resisting medium.

THEOREM 5. *If a particle starts from rest in any non-vanishing field of force, the first curvature of the trajectory and that of the line of force are in the ratio* $1 : 3 + 2R_0/f_0$.

9. If $\bar{\kappa}_1 = 0$ then κ_1 is also equal to zero. The trajectory and the line of force have contact of order 2 by (3·3) and hence $\bar{f}_0^{r(2)} = Q_0^{r(2)}$.

$$\left.\begin{aligned} \bar{f}_0^{r(2)} &= T''_0 \lambda_0^r + \kappa'_1 (5f_0 + 4R_0)\lambda_1^r \\ Q_0^{r(2)} &= f''_0 \bar{\lambda}_0^r + \bar{\kappa}'_1 f_0 \bar{\lambda}_1^r \end{aligned}\right\} \tag{9·1}$$

Equating the coefficients of λ_1^r and $\bar{\lambda}_1^r$ we get

$$\kappa'_1 (5f_0 + 4R_0) = \bar{\kappa}'_1 f_0$$

and hence at O

$$\kappa'_1 : \bar{\kappa}'_1 = 1 : 5 + \frac{4R_0}{f_0} \tag{9·2}$$

Now we shall prove the following Theorem by induction.

THEOREM 6. *If a particle starts from rest at a point O and if* $\bar{\kappa}_1, \bar{\kappa}'_1 \ldots \bar{\kappa}_1^{(p-2)}$ *are all zero, then* $\kappa_1, \kappa'_1 \ldots \kappa_1^{(p-2)}$ *are also zero, so that the trajectory and the line of force have contact of order p. Then*

$$\kappa_1^{(p-1)} : \bar{\kappa}_1^{(p-1)} = 1 : (2p+1) + \frac{2pR_0}{f_0}.$$

Following the usual mode of proof, by assuming that this is true for values up to $(p-1)$ we are led to the condition that the trajectory and the line of force have contact of order p. Then $\bar{f}_0^{r(p)} = Q_0^{r(p)}$.

$$\begin{aligned}\bar{f}^{r(p)} = {} & \lambda_0^r[T^{(p)} + A_m \kappa_1^{(m)}] \\ & + \lambda_1^r\Big[B_m \kappa_1^{(m)} + \kappa_1^{(p)} v^2 + \kappa_1^{(p-1)}\{(2p+1)T + 2pR\}\Big] \\ & + \lambda_2^r[C_m \kappa_1^{(m)} + p\kappa_2 \kappa_1^{(p-1)} v^2] \\ & + \lambda_s^r[D_m^s \kappa_1^{(m)}] \end{aligned} \tag{9·3}$$

$$(m=0, 1, \ldots p-2), \ (s=3, 4, \ldots p+1), \ D_m^n = 0, \ m>1$$
$$\text{and } D_m^{n+1} = 0, \ m>0.$$

Also

$$Q^{r(p)} = \bar{\lambda}_0^r[f^{(p)} + \bar{A}_m \bar{\kappa}_1^{(m)}] + \bar{\lambda}_1^r[\bar{B}_m \bar{\kappa}_1^{(m)} + f\bar{\kappa}_1^{(p-1)}] + \bar{\lambda}_s^r[\bar{C}_m^s \bar{\kappa}_1^{(m)}] \tag{9·4}$$

$$(m=0, 1, \ldots p-2), \ (s=2, 3, \ldots p), \ \bar{C}_m^{n-1} = 0, \ m>1$$
$$\text{and } \bar{C}_m^n = 0, \ m>0.$$

Since $\qquad \kappa_1^{(m)} = 0 = \bar{\kappa}_1^{(m)} \ (m=0, 1, \ldots p-2)$

we have

$$\left.\begin{aligned} \bar{f}_0^{r(p)} &= T^{(p)}\lambda_0^r + [(2p+1)T_0 + 2pR_0]\kappa_1^{(p-1)}\lambda_1^r \\ Q_0^{r(p)} &= f_0^{(p)}\bar{\lambda}_0^r + f_0 \bar{\kappa}_1^{(p-1)}\bar{\lambda}_1^r \end{aligned}\right\} \tag{9·5}$$

As usual, equating the coefficients of λ_1^r and $\bar{\lambda}_1^r$ we get

$$\kappa_1^{(p-1)} : \bar{\kappa}_1^{(p-1)} = 1 : (2p+1) + \frac{2pR_0}{f_0}. \tag{9·6}$$

10. For a particle projected with a velocity v in the direction of the line of force we have from equations (9·3) and (9·4), and the fact that R occurs as a coefficient of $\kappa_1^{(p-1)}$ which vanishes, that $\bar{f}_0^{r(p)}$ and $Q_0^{r(p)}$ reduce to those in (7·3). Hence Theorems 3 and 4 hold good even in a resisting medium. Hence we note that the initial curvature of a free particle is influenced by the resistance of the medium when $R_0 \neq 0$ whereas, if the particle is projected with a velocity v, its presence is not felt as far as the initial curvature and its derivatives are concerned.

A PROPERTY OF THE ZEROS OF THE SUCCESSIVE DERIVATIVES OF INTEGRAL FUNCTIONS

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[Received 2 July 1937]

1. The following result has been proved by Takenaka:*

THEOREM 1. *Let $f(z)$ be an integral function of order one and type not exceeding σ. Let $[\alpha_n]$ be a sequence such that*

$$\lim_{n\to\infty} |\alpha_n| = L < \tfrac{1}{\sigma} \log 2. \tag{1}$$

Let $f^{(n)}(\alpha_n)=0$, $n=0, 1, 2 \ldots [f^{(o)}(z)\equiv f(z)]$. Then $f(z)\equiv 0$.

1.1. The function $\sin z - \cos z$ shows that log 2 in (1) cannot be replaced by any number greater than $\pi/4$, and it has been conjectured† that this might be the best possible result and this conjecture has been proved true‡ when the numbers $[\alpha_n]$ are real. In this paper I show that when $f(z)$ is odd or even, log 2 in (1) can be replaced by $\log(2+\sqrt{3}) > 1 > \pi/4$; the function $\sin z$ shows that probably $\pi/2$ is the "best possible constant" in the case of odd or even functions. The method used closely resembles that employed by Takenaka to prove Theorem 1. The result can also be proved by a modification of the proof given by J. M. Whittaker§ for Theorem 1.

2. We prove

THEOREM 2. *Let $f(z)$ be an even or odd integral function of order one and type not exceeding σ. Let $[\alpha_n]$ be a sequence such that*

$$\overline{\lim_{\omega\to\infty}} |\alpha_n| = L < \frac{\log(2+\sqrt{3})}{\sigma}. \tag{2}$$

If $f^{(n)}(\alpha_n)=0$, $n=0, 1, 2, \ldots$, then $f(z)\equiv 0$.

* *Proc. Physico. Math. Soc.* Japan, 14 (1932), 529-42.

† *Cf.* J. M. Whittaker, *Interpolatory Function Theory*, Camb. Tract, No. 33, p. 45.

‡ Schoenberg, *Trans. Amer. Math. Soc.*, 40 (1936), 12-23.

§ loc. cit.

2.1. Theorem 2 is derived from

THEOREM 3. *Let $f(z)$ be an even integral function of order one and type $\sigma<1$. Let $[\alpha_{2n}]$, $n=0, 1, \ldots$ be such that*

$$|\alpha_{2n}| \leqslant L < \log(2+\sqrt{3}). \tag{3}$$

Then, for all finite z,

$$f(z)=\sum_{0}^{\infty} f^{(2n)}(\alpha_{2n})\, p_{2n}(z). \tag{4}$$

where

$p_0(z)\equiv 1$, $p_{2n}(z)$

$$=\int_{\alpha_0}^{z} dt_1 \int_0^{t_1} dt_2 \int_{\alpha_2}^{t_2} dt_3 \int_0^{t_3} dt_4 \ldots \int_{\alpha_{2n-2}}^{t_{2n-2}} dt_{2n-1} \int_0^{t_{2n-1}} dt_{2n} \int_{\alpha_{2n}}^{t_{2n}} dt_{2n+1} \tag{5}$$

for $n \geqslant 1$.

2.2. To prove Theorem 3 we need the following

LEMMA. *Let $[\alpha_{2n}]$, $n=0, 1, \ldots$, satisfy the condition (3). Let $\phi(z)$ be an even function regular in $|z|\leqslant R(>1)$. Then*

$$\phi(z)=\sum_{0}^{\infty} c_{2n} z^{2n} \cosh \alpha_{2n} z, \tag{6}$$

the series on the right side of (6) converging absolutely and uniformly for $|z|\leqslant r<1$.

PROOF. Let

$$\phi(z)=\sum_{0}^{\infty} a_{2n} z^{2n}. \tag{7}$$

Comparing (6) and (7) formally, we get

$$\left.\begin{aligned} &c_0=a_0,\\ &c_{2n}=a_{2n}-c_{2n-2}\frac{(\alpha_{2n-2})^2}{2!}-\ldots-\frac{c_0\alpha_0^{2n}}{2n!},\ n\geqslant 1. \end{aligned}\right\} \tag{8}$$

Now, given $[\alpha_{2n}]$, $[a_{2n}]$, $n=0, 1, 2, \ldots$, suppose that the $[c_{2n}]$ are calculated from (8). If for this $[c_{2n}]$, the series (6) converges absolutely and uniformly in some circle $|z|<\rho>0$, we get by (7) and (8),

$$\phi^{(2n)}(0)=\psi^{(2n)}(0).$$

where $\psi(z)$ denotes the sum of the series (6) in $|z|<\rho$. Hence $\phi(z)\equiv\psi(z)$. Therefore, if it is shown that the series in (6) converges absolutely and uniformly for $|z|\leqslant r<1$, the Lemma would be proved. First we note that, in virtue of (3),

$$\lim_{n\to\infty} |z^{2n}\cosh \alpha_{2n}z|^{\frac{1}{2n}}=|z|. \tag{9}$$

Next, we show that for a properly chosen k, we must have

$$|c_{2s}|\leqslant k, \tag{10}$$

for $s=0, 1, 2,\ldots$ Suppose that for some k, (10) holds for $s=0, 1, 2,\ldots,n-1$. Then (8) gives

$$\begin{aligned}|c_{2n}|&\leqslant|a_{2n}|+k\left[\frac{L^2}{2\,!}+\frac{L^4}{4\,!}+\ldots+\frac{L^{2n}}{2n\,!}\right]\\ &\leqslant|a_{2n}|+k\,(\cosh L-1).\end{aligned} \tag{11}$$

Now, if $M(R)=\max\limits_{|z|\leqslant R}|\phi(z)|$, we have

$$|a_{2n}|\leqslant\frac{M(R)}{R^{2n}}\to 0, \tag{12}$$

as $n\to\infty$, since $R>1$. Since $L<\log(2+\sqrt{3})$, we get

$$\cosh L-1<\cosh\log(2-\sqrt{3})-1=1. \tag{13}$$

Combining (11), (12) and (13), we conclude that there exists a constant k depending only on L, R and $M(R)$, such that the relations

$$|c_{2s}|\leqslant k,\ s=0, 1,\ldots n-1$$

involve

$$|c_{2n}|\leqslant k.$$

Hence by induction, (10) holds for a properly chosen k and the Lemma follows from (9) and (10).

2.3. *Proof of Theorem* 3. By (5), $p_{2n}^{(2s-1)}(0)=0$, $s=1,\ldots n$. Therefore we can write

$$p_{2n}(z)=\sum_{\nu=0}^{n}\lambda_{2\nu}^{(n)}\frac{z^{2\nu}}{2\nu\,!}. \tag{14}$$

Let

$$\pi_{2n}(z)=\sum_{\nu=0}^{n}\lambda_{2\nu}^{(n)}z^{2\nu}. \tag{15}$$

Then if Γ denotes any circle round $z=0$,

$$p_{2n}(z)=\frac{1}{2\pi i}\int_{\Gamma}\frac{\pi_{2n}(s)}{s}\cosh\frac{z}{s}ds. \tag{16}$$

From (5) and (16) we get

$$\frac{1}{2\pi i}\int_{\Gamma}\frac{\pi_{2n}(s)}{s^{2\nu+1}}\cosh\frac{\alpha_{2\nu}}{s}ds=\left.\begin{matrix}0, \ \nu\neq n,\\ 1, \ \nu=n.\end{matrix}\right\} \tag{17}$$

For a given z, $\cosh xz$ is regular for all x; hence putting $x=1/s$ and applying the lemma we get

$$\cosh\frac{z}{s}=\sum_{0}^{\infty}\frac{c_{2n}(z)}{s^{2n}}\cosh\frac{\alpha_{2n}}{s}, \tag{18}$$

which converges absolutely and uniformly for $|s|\geqslant d>1$. Taking $|s|=d$ for Γ in (16) and using (17), we get from (18), that

$$c_{2n}(z)=p_{2n}(z),\ n=0, 1, 2,\ldots$$

so that, for $|s|>1$,

$$\cosh\frac{z}{s}=\sum_{0}^{\infty}\frac{p_{2n}(z)}{s^{2n}}\cosh\frac{\alpha_{2n}}{s}. \tag{19}$$

Now let

$$f(z)=\sum_{0}^{\infty}\frac{b_{2n}}{2n!}z^{2n}. \tag{20}$$

and

$$\phi(z)=\sum_{0}^{\infty}b_{2n}z^{2n}. \tag{21}$$

Since $\overline{\lim_{n\to\infty}}|b_{2n}|^{\frac{1}{2n}}=\sigma<1$, $\phi(z)$ is regular in some circle $C:|z|\leqslant d>1$. Taking C for Γ, we get from (19), (20) and (21),

$$\begin{aligned} f(z)&=\frac{1}{2\pi i}\int_{C}\frac{\phi(s)}{s}\cosh\frac{z}{s}ds\\ &=\sum_{n=0}^{\infty}p_{2n}(z)\left\{\frac{1}{2\pi i}\int_{C}\frac{\phi(s)}{s^{2n+1}}\cosh\frac{\alpha_{2n}}{s}ds\right\}\\ &=\sum_{0}^{\infty}p_{2n}(z)f^{(2n)}(\alpha_{2n}),\end{aligned}$$

which proves the Theorem.

2.4. *Proof of Theorem* 2. Suppose first that

$$\left.\begin{array}{c} f(z) \text{ is even, } \sigma<1, \\ |\alpha_{2n}\leqslant L<\log(2+\sqrt{3}),\ n=0,1,2,\ldots \end{array}\right\} \qquad (22)$$

In this case the Theorem follows at once from the relation (4) since the hypothesis of Theorem 3 holds and

$$f^{(2n)}(\alpha_{2n})=0,\ n=0,1,2,\ldots.$$

Next suppose

$$\left.\begin{array}{c} f(z) \text{ is even} \\ |\alpha_{2n}|\leqslant L<\dfrac{\log(2+\sqrt{3})}{\sigma},\ n=0,1,2\ldots \end{array}\right\} \qquad (23)$$

while $f(z)$ is of type not exceeding σ. Let

$$g(z)=f\left(\frac{z}{\sigma+\varepsilon}\right),\ \beta_{2n}=(\sigma+\varepsilon)\alpha_{2n}$$

where $\varepsilon>0$. Then the type of $g(z)$ does not exceed $\frac{\sigma}{\sigma+\varepsilon}<1$ while $g^{(2n)}(\beta_{2n})=0$; moreover

$$|\beta_{2n}|\leqslant(\sigma+\varepsilon)L<\frac{\sigma+\varepsilon}{\sigma}\log(2+\sqrt{3})<\log(2+\sqrt{3}),$$

if $\varepsilon>0$ is small enough. Hence $g(z)$ satisfies a hypothesis of the type (22) so that $g(z)\equiv0$ which involves $f(z)\equiv0$. Next suppose that

$$\overline{\lim}\ |\alpha_n|=L<\frac{\log(2+\sqrt{3})}{\sigma}$$

while $f(z)$ is even and of type not exceeding σ. Let $\eta>0$ be such that

$$|\alpha_n|\leqslant L+\eta<\frac{\log(2+\sqrt{3})}{\sigma} \text{ for } n\geqslant 2n_0.$$

Then $f^{(2n_0)}(z)=g(z)$ satisfies (23) with $\beta_n=\alpha_{n+2n_0}$. Hence $f^{(2n_0)}(z)\equiv0$ and since $f^{(2s)}(\alpha_{2s})=0$, $s=0,1,\ldots n_0-1$ while $f^{(2s-1)}(0)=0$, $s=1,\ldots n_0$, we get $f(z)\equiv0$. Hence, Theorem 2 holds when $f(z)$ is even. If $f(z)$ is odd, then $g(z)=f'(z)$ is even and satisfies the conditions of Theorem 2 with $\beta_n=\alpha_{n+1}$. Hence $f'(z)\equiv0$; therefore $f(z)\equiv0$ since it is odd. This proves Theorem 2.

3. For functions regular in a finite circle the following result has been proved by Kakeya.*

* *See* the reference in footnote 1, p. 125.

THEOREM 4. *Let $f(z)$ be regular in $|z|<R$. Let $[\alpha_n]$ be a sequence such that*

$$\overline{\lim_{n\to\infty}}\, n|\alpha_n|=L<R\log 2.$$

Then $f^{(n)}(\alpha_n)=0$, $n=0, 1, 2, \ldots$, involves $f(z)\equiv 0$.

3.1. By modifying Kakeya's method, or using the functions

$$\tfrac{1}{2}z^{2n}\left[\frac{1}{(1-\alpha_{2n}z)^{2n+1}}+\frac{1}{(1+\alpha_{2n}z)^{2n+1}}\right]$$

instead of $z^{2n}\cosh\alpha_{2n}z$ and proceeding as in the proof of Theorem 2 we can establish

THEOREM 5. *Let $f(z)$ be an even or odd function regular in $|z|<R$. Let $[\alpha_n]$ be such that*

$$\overline{\lim_{n\to\infty}}\, n|\alpha_n|=L<R\log(2+\sqrt{3}).$$

Then $f^{(n)}(\alpha_n)=0$ involves $f(z)\equiv 0$.

3.2. It may be noted that in Theorems 2 and 5 it is sufficient to suppose that

$$f^{(2n)}(\alpha_{2n})=0,\ n=0, 1, 2\ldots$$

when $f(z)$ is even and

$$f^{2n+1}(\alpha_{2n+1})=0,\ n=0, 1, 2,\ldots.$$

when $f(z)$ is odd, since the remaining derivatives in the respective cases vanish at $z=0$.

The Indian Mathematical Society.

Founded in 1907 for the Advancement of Mathematical Study and Research in India.

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Vol. II. No. 7.

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